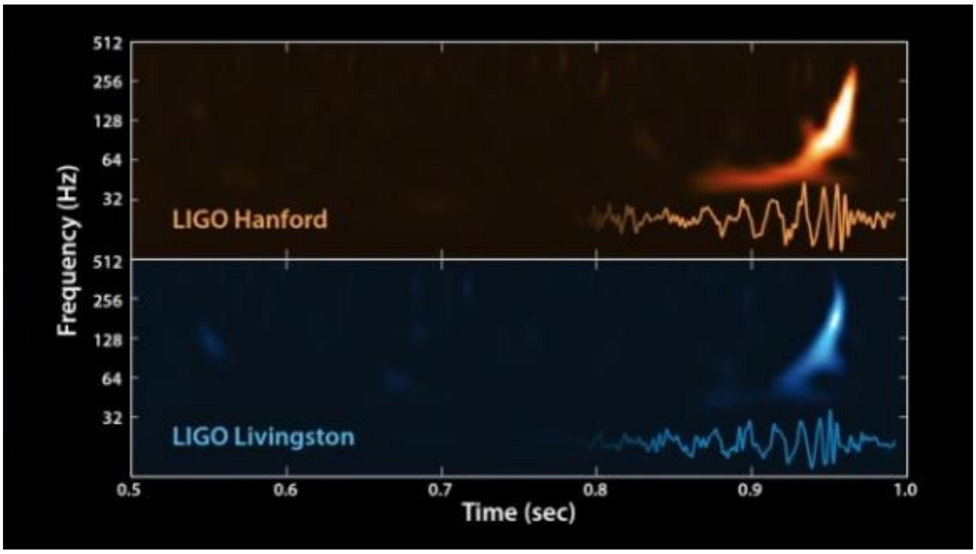
The Three Italians that paved Einstein's path to General Relativity and the origins of Differential Geometry.

Pietro Fré (University of Torino & Embassy of Italy in Moscow)

The first detection of gravitational waves, occurred in the centennial year of General Relativity, was not only a spectacular confirmation of Einstein's Theory, but also the occasion of seeing the differential equations encoded in the Ricci tensor at work. Indeed this is what is ultimately encoded in the the signal registered by the two Ligos. Thus, in this exciting year 2016, while we wait for the new data that the three interferometers will provide us with, just as they start again on duty this fall, it is most appropriate to review the interesting conceptual path that, crossing through the entire history of XIX century Mathematics, led to the merging of Physics into Geometry finally operated by Einstein at the dawn of the XX century. Within the framework of this tale, it is not only appropriate but long due and obligatory to recall and commemorate the extraordinary role played by three great Italians, namely ***Ricci Curbastro, Bianchi and Levi Civita***. Of the last and youngest one among these three, Levi Civita, Einstein used to say that, apart from spaghetti, he was the most important thing that existed in Italy.

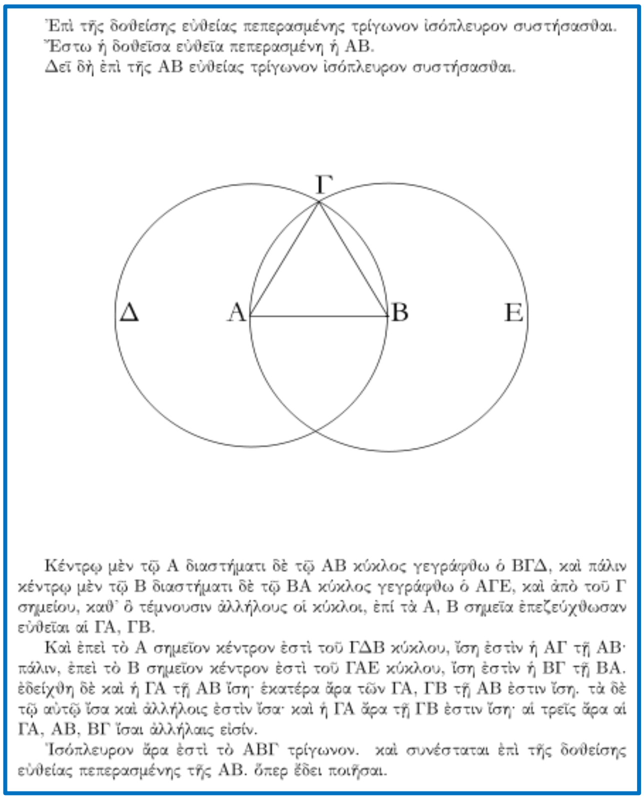


Indeed in order to arrive at the conception of Space-Time that underlies Einstein's Theory, three conceptual revolutions were needed:

1. Euclidian geometry, that had been considered by Kant *an apriori truth at the fundament of all sensorial experience*, had to be dethronized, making it possible to consider non-euclidian signatures of what was later named a metric.
2. Cartesian coordinates had to be detronized, in favor of intrinsic coordinates able to describe manifolds of arbitrary curvature. A new tensor calculus was needed to treat new mathematical entities that soon were turned by Einstein into physical ones.
3. Group Theory ought to be developed in order to formalize the notion of general coordinate transformations and to understand the symmetries of various geometries.

Nicolai Ivanovich Lobachevsky (1793-1856)

The first revolution was initiaded by the Russian mathematician Nicolai Ivanovich Lobachevsky (1793-1856) who introduced the first model of a non-Euclidian geometry in 1826. It was continued by Eugenio Beltrami who, in 1868, provided explicit representations of Lobachevsky geometry on curved surfaces and by Poincaré, who encoded such a geometry into a two-manifold with a specific metric. The second and the third revolutions were respectively initiated by Gauss in 1828 and by Galois in 1832. The complicated conceptual history that followed these events and culminated in the founding, by ***Ricci, Levi-Civita*** and ***Bianchi***, of a new mathematical discipline, namely ***Differential Geometry***, is the tale told in the next lines. We begin by reporting the words that open Levi Civita and Ricci's paper of 1899. They are particularly inspiring in view of what was to follow:



Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.
Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.
Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.

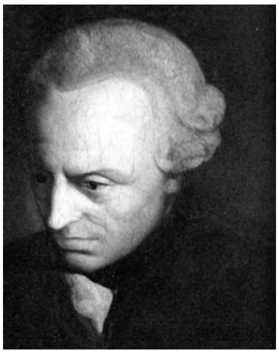
Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. ὅπερ ἔδει ποιῆσαι.

The initial page in the Greek original of Euclid's Elements.

M. Poincaré a écrit que dans les Sciences mathématiques une bonne notation a la même importance philosophique qu'une bonne classification dans les Sciences naturelles. Évidemment, et même avec plus de raison, on peut en dire autant des méthodes, car c'est bien de leur choix que dépend la possibilité de forcer (pour nous servir encore des paroles de l'illustre géomètre français) une multitude de faits sans aucun lien apparent à se grouper suivant leurs affinités naturelles. (Ricci & Levi-Civita)

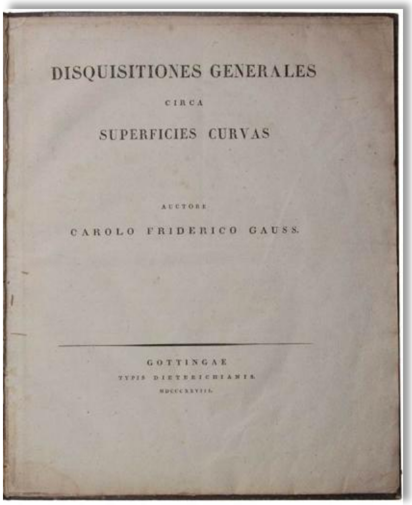


Immanuel Kant (1724 - 1804)

Eugenio Beltrami (1836 - 1900)

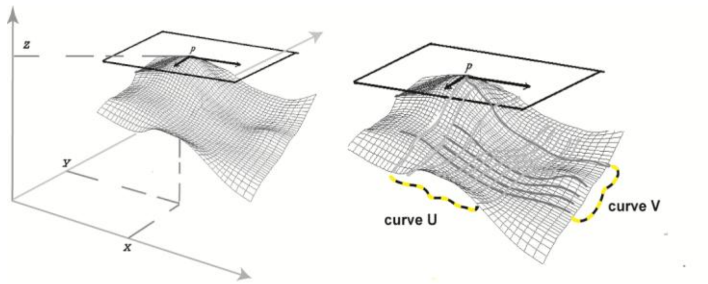
Henri Poincaré (1854-1912)

Figura 1 Carl Friedrich Gauss (1777 -- 1855). Gauss, the *King of Mathematicians*, was Professor at the University of Gottingen for many decades up to the very end of his long life. His contributions to all fields of Mathematics were enormous and most profound.



Gauss introduces intrinsic geometry and curvilinear coordinates in 1828

The first appearance of a metric is in the 1828 essay of Gauss on curved surfaces. Written in Latin, the *Disquisitiones Generales circa superficies curvas* contains the major revolutionary step forward that was necessary to overcome the precincts of Euclidian geometry and found a new differential science of spaces able to treat both flat and curved ones. Up to Gauss' paper, Geometry was either formulated abstractly in terms of Euclidian axioms or analytically in terms of Cartesian coordinates. By Geometry it was meant the study of global properties of plane figures like triangles, squares and other polygons, or solids like the regular polyhedra. All such objects were conceived as immersed in an external space where it was implicitly assumed that one could always define the absolute distance d(A,B) between any two given points A and B. Distance is the basic brick of the whole Euclidian building and it is calculated as the length of the segment with end-points in A and B, lying on the unique straight line which goes through any such pair of distinct points. Curved surfaces were obviously known before Gauss, yet their shape and properties were conceived only through their immersion in three-dimensional space, considered unique and absolute, as pretended by Immanuel Kant who promoted Euclidian geometry to an a priori truth lying at the basis of any sensorial experience. Gauss revolutionary starting point was that of reformulating the geometrical study of surfaces from an intrinsic rather than extrinsic viewpoint. He wondered how a little being, confined to live on the surface, might have perceived the geometry of his world. Rather than viewing the global shape of the surface M, inaccessible to his observations, the little creature would have explored its local properties in the vicinity of a point p of M. In order to study curved surfaces in these terms, Gauss understood that it was necessary to abandon Cartesian coordinates as a system of point identification. Gauss was the first to grasp the notion of curvilinear coordinates and invented *Gaussian coordinates*. A very simple but revolutionary idea. By introducing curvilinear Gaussian coordinates, the King of Mathematicians freed the study of surfaces from their immersion in the external Euclidian space but he immediately had to cope with a new fundamental problem. Having abolished from the list of one's mathematical instruments the straight line segments that join any two points A and B of the surface M, how can we calculate their distance? The great intuitions of Gauss were the *tangent plane* and the linear element ds^2,namely the metric. The problem addressed by Gauss was to give an answer to the following question: Can we define the length of any curve departing



Bernhard Riemann (1826 -- 1866). Riemann's dissertation was published posthumous by Dedekind on the ABHANDLUNGEN DER KÖNIGLICHEN GESELLSCHAFT DER WISSENSCHAFTEN ZU GÖTTINGEN

from p and arriving at q, both in M, in terms of data completely intrinsic to the surface. Gauss' answer was positive and based on the change of perspective at the basis of the new *differential geometry*. Let us reformulate the initial question whether we might define the absolute distance between two arbitrary points A,B of the surface M, adding the extra condition that A and B should be only infinitesimally apart from each other. Analytically this means that if the Gaussian coordinates of A are *(u,v)*, then those of B should be *(u+du,v+dv)* where *du* e *dv* are infinitesimal. Gauss crucial observation is that a very small portion of the surface M, around any point p, can be approximated by a portion of the tangent plane to the surface at the point p. The square length of the segment joining A and B, named in modern notation ds^2, was expressed by Gauss as a quadratic form in the differentials (*du, dv)*, namely :

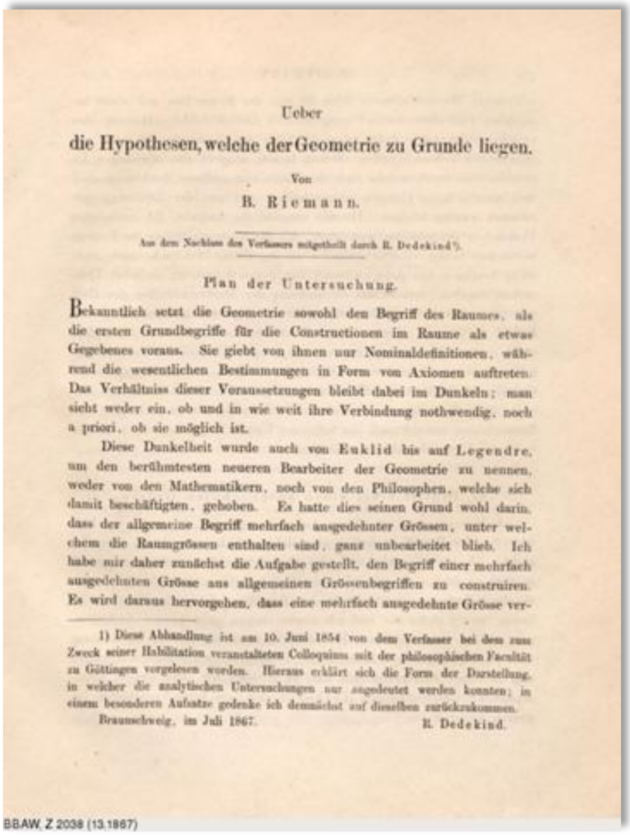
$$ds^2 = F(u,v)\, du^2 + G(u,v)\, dv^2 + H(u,v)\, dudv.$$

Written in 1828 this formula provided the first example of a Riemannian metric, although Riemann was at that time only a two-year old child.

Bernhard Riemann introduces n-dimensional metric manifolds in 1854

The name of Riemann is associated in Mathematics with so many different and fundamental objects that the contemporary student is instinctively led to think about the scientific production of this giant of human thought as composed by a countless number of papers, books and contributions. Actually the entire corpus of Riemann's works is constituted only by 225 pages distributed over 11 articles published during the life-time of their author to which one has to add the 102 pages of the 4 posthumous publications.

Among the latter there are the 16 pages of the *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen* which, in 1854, was debated by the candidate in front of the Göttingen Faculty of Philosophy as *Habilitationsschrift*. The habilitation to teach courses was the traditional first step in the academic career foreseen by most European universities, all over their very long history. In XIX century Germany the procedure to access habilitation consisted of the writing of a dissertation on a topic chosen by the Faculty from a list of three proposed by the candidate. Typical time allowed for the preparation of such a dissertation was a couple of months and in the case of Riemann it amounted to exactly seven weeks. Obsessed the whole of his short life by extreme poverty and by a very poor health, that eventually led him to death from pulmonary consumption at the quite young age of thirty-nine, the shy and meek Bernhard Riemann, who was nonetheless quite conscious of his own talents, had already profoundly impressed Gauss with his diploma thesis. Written in 1851 and entitled *Grundlagen fur eine allgemeine Theorie der Functionen einer Veränderlichen complexen Grösse* which can be translated as *Principles of a General Theory of the Functions of one complex variable*, Riemann's thesis was completely new and contained all the essentials of the theory of analytic functions as it is taught up to the present day in most universities of the world. Quite openly Gauss told his young student that for many years he had cheered the plan of writing a similar essay on that very topics, yet now he would refrain from doing so since everything relevant to that province of thought had already been said by Riemann.


Ueber

die Hypothesen, welche der Geometrie zu Grunde liegen.

Von

B. Riemann.

Aus dem Nachlass des Verfassers mitgetheilt durch R. Dedekind[1].

Plan der Untersuchung.

Bekanntlich setzt die Geometrie sowohl den Begriff des Raumes, als die ersten Grundbegriffe für die Constructionen im Raume als etwas Gegebenes voraus. Sie giebt von ihnen nur Nominaldefinitionen, während die wesentlichen Bestimmungen in Form von Axiomen auftreten. Das Verhältniss dieser Voraussetzungen bleibt dabei im Dunkeln; man sieht weder ein, ob und in wie weit ihre Verbindung nothwendig, noch a priori, ob sie möglich ist.

Diese Dunkelheit wurde auch von Euklid bis auf Legendre, um den berühmtesten neueren Bearbeiter der Geometrie zu nennen, weder von den Mathematikern, noch von den Philosophen, welche sich damit beschäftigten, gehoben. Es hatte dies seinen Grund wohl darin, dass der allgemeine Begriff mehrfach ausgedehnter Grössen, unter welchem die Raumgrössen enthalten sind, ganz unbearbeitet blieb. Ich habe mir daher zunächst die Aufgabe gestellt, den Begriff einer mehrfach ausgedehnten Grösse aus allgemeinen Grössenbegriffen zu construiren. Es wird daraus hervorgehen, dass eine mehrfach ausgedehnte Grösse ver-

1) Diese Abhandlung ist am 10. Juni 1854 von dem Verfasser bei dem zum Zweck seiner Habilitation veranstalteten Colloquium mit der philosophischen Facultät zu Göttingen vorgelesen worden. Hieraus erklärt sich die Form der Darstellung, in welcher die analytischen Untersuchungen nur angedeutet werden konnten; in einem besonderen Aufsatze gedenke ich demnächst auf dieselben zurückzukommen.

Braunschweig, im Juli 1867. R. Dedekind.

BBAW, Z 2038 (13.1867)

When three years later Riemann presented to the Göttingen Faculty his three proposals for the theme of his own *Habilitationsschrift*, two choices were in fields where the young mathematician felt quite confident, while the third, with some hesitation, was just added in order to complete the triplet and with the secret hope that it would be immediately discarded by the academic committee as something too philosophical and ill defined. The third proposed title was *Grundlagen der Geometrie*, namely the Principles of Geometry. Remembering the talents of the young *Herr Riemann*, Gauss was fascinated by the idea of giving him precisely such a challenging subject as the Foundations of Geometry to see what he might come up with it. The King of Mathematicians persuaded the Faculty to make such a choice and the poor Bernhard was dismayed by the news. He wrote to his father, a poor Lutheran minister, about his concerns on this matter but he also expressed him his confidence that he would not come too late and that his merits as an independent researcher would be appreciated.

Riemann had accepted the challenge and in seven weeks he produced such a masterpiece of Mathematics and Philosophy as the *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*, that is *About the Hypotheses lying at the Foundations of Geometry*. With an unparalleled clarity of mind, Riemann began his essay with a profound criticism of the traditional approach to Geometry, refusing the Kantian dogma that this latter is an a-priori datum and rather inclining to the idea that which geometry is the actual one of Physical Space should be determined from experience. He said: *It is known that geometry assumes, as things given, both the notion of space and the first principles of constructions in space. She gives definitions of them which are merely nominal, while the true determinations appear in the form of axioms. The relation of these assumptions remains consequently in darkness; we neither perceive whether and how far their connection is necessary, nor a priori, whether it is possible. From Euclid to Legendre (to name the most famous of modern reforming geometers) this darkness was cleared up neither by mathematicians nor by such philosophers as concerned themselves with it*[1].

After stating this two-thousand year old stalemate, Riemann proceeded to diagnose its cause. Explicitly he said: The reason of this is doubtless that the general notion of multiply extended magnitudes (in which space-magnitudes are included) remained entirely unworked. I have in the first place, therefore, set myself the task of constructing the notion of a multiply extended magnitude out of general notions of magnitude. It will follow from this that a multiply extended magnitude is capable of different measure-relations, and consequently that space is only a particular case of a triply extended magnitude. In contemporary language the *multiply extended magnitudes*[2] were simply the *manifolds* and the *measure*

[1]The translation of Riemann's essay from German into English was done by William Clifford.

[2]In the original German text of Riemann these were named *mehrfachausgedehnter Grossen*. In modern scientific German the notion of manifolds is referred to *as mannigfaltigkeiten*.

relations are just the metrics introduced for the first time by Gauss. Following the new road opened by Gauss with the *Disquisitiones*, Riemann introduced n-extended manifolds whose points are labeled by n rather than two curvilinear coordinates x^i and introduced the line element as a generic symmetric quadratic form in the differentials of these coordinates

$$ds^2 = g_{ij}(x)\, dx^i dx^j.$$

The coefficients of this quadratic form $g_{ij}(x)$ were later known as the Riemannian *metric tensor.* Riemann grasped the main point, namely that the geometry of manifolds is encoded in the possible metric tensors or *measure relations*, as he called them, and made the following bold statement: *Hence flows as a necessary consequence that the propositions of geometry cannot be derived from general notions of magnitude, but that the properties which distinguish Space from other conceivable triply extended magnitudes are only to be deduced from experience. Thus arises the problem, to discover the simplest matters of fact from which the measure-relations of space may be determined; a problem which from the nature of the case is not completely determinate, since there may be several systems of matters of fact which suffice to determine the measure-relations of space*.

In other words, the young genius was aware that the same manifold could support quite different metrics and thought that this applied in particular to Space, i.e. to the 3-dimensional physical world of our sensorial experience. He posed himself the question which should be the metric of Space and came to the conclusion that such a question could only be answered through experiment. This amounted to say that the geometry of the world is a matter of Physics and not of a priori Philosophy or Mathematics. Such a sentence of Riemann must have influenced Einstein quite deeply. Indeed the final outcome of Einstein Theory of Relativity is that the geometry of space-time is dynamically determined by its matter content through Einstein field equations. In considering such a question as what is the preferred metric to be selected for a given manifold, Riemann formulated the basic problem of *invariants*. The matter of facts[3] to which he alluded are the intrinsic properties encoded in a given metric tensor namely its invariants and he formulated the problem of determining, for instance, the minimal complete number of invariants able to select Euclidian geometry. In his quest for these invariants he came to the notion of the Riemann curvature tensor that he outlined in his very dissertation. As we already recalled, Riemann died young and had no time to develop the new theory of differential geometry that he had founded. Yet he had the time to come to Italy and, through his contact with the Scuola Normale di Pisa and the research group of Enrico Betti, whom he deeply admired, to plant the seeds of the absolute differential calculus in the Italian Peninsula where, later, they were strongly developed by Gregorio Ricci Curbastro and Tullio Levi Civita.

The Absolute Differential Calculus of Ricci Curbastro and Levi Civita (1899)

The primary concern of the new differential geometry, founded by Riemann as a generalization of Gauss work on surfaces, was that of defining the length of curves on arbitrary manifolds. This leads to the notion of the metric. Once the metric is established, a natural way arises of transporting vectors along any given curve. We can say that a vector is parallel-transported along an arc of curve if the angle between the transported vector and the tangent vector to the curve remains constant throughout the entire transport. The metric connection is that infinitesimal displacement of a vector X along the direction singled out by another one Y which is so defined as to fulfill the property of preserving

Elwin Bruno Christoffel (1929 – 1918)

[3] *EinfachstenThatsachen*in the original German text.

angles. It was first conceived by Christoffel. Elwin Bruno Christoffel was born in 1829 in Montjoie, near Aachen, that was renamed Monschau in 1918. After attending secondary schools in Cologne, he enrolled at the University of Berlin, where he had such teachers as Eisenstein and Dirichlet. Particularly the latter is duly considered his master. Christoffel's doctor dissertation, dealing with the motion of electricity in homogeneous media was defended in 1856, just two years after Riemann's presentation of the *Ueber die Hypothesen*. Having spent a few years out of the academic world, Christoffel returned to Mathematics in 1859, obtaining his *habilitation* from Berlin University. In the following years he was professor at the Polytechnic of Zurich, at the newly founded Technical University of Berlin and finally at the University of Strasbourg which had become German after the defeat of Napoleon III in the 1870 war. Although he wrote papers on several different topics like potential theory, differential equations, conformal mappings, orthogonal polynomials and still more, the most relevant and influential of Christoffel's contributions with the furthest reaching consequences was his invention of the three-index symbols that bear his name:

$$\left\{ \begin{matrix} \lambda \\ \mu\,\nu \end{matrix} \right\} \equiv \tfrac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

Defined in terms of a metric $\mathbf{g}_{\mu\nu}$ and of its inverse, Christoffel symbols are the first example of a set of connection coefficients, actually those of the Levi-Civita connection, that preserves angles along the parallel transport it defines. Christoffel symbols are the key ingredients in the definition of the covariant derivative of a tensor, in particular of a vector. The word tensor was introduced for the first time by Hamilton in 1846, but tensor calculus was developed around 1890 by Gregorio Ricci Curbastro under the title of *absolute differential calculus* and was made accessible to mathematicians by the publication of Tullio Levi Civita's 1900 classic text of the same name, originally written in Italian, then republished in French with Ricci.

Méthodes de calcul différentiel absolu et leurs applications.

Par

M. M. G. Ricci et T. Levi-Civita à Padoue.

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Algorithme du calcul différentiel absolu.

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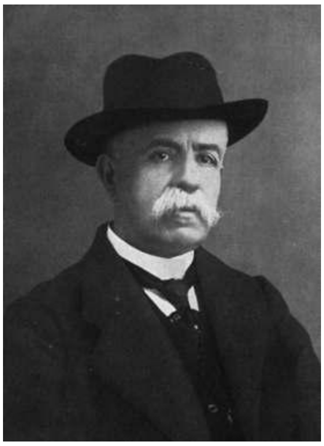
Applications analytiques.

Gregorio Ricci Curbastro was son in an aristocratic family of Lugo di Romagna. On the house where he was born in 1853 (one year before Riemann presented his famous dissertation) there stands a plate with the following words: *Diede alla scienza il calcolo differenziale assoluto, strumento indispensabile per la teoria della relativita' generale, visione nuova dell'universo*. He began his studies at Rome University but he continued them at Scuola Normale di Pisa and finally graduated from the University of Padova in 1875. As his younger friend Luigi Bianchi, born in Parma in 1865 and also student of Scuola Normale, in the Pisa years he was deeply influenced by the teaching of Ulisse Dini and Enrico Betti, the founder of modern topology. Through Betti, both



Gregorio Ricci Curbastro (1853 -1925)

Ricci and Bianchi captured the seeds of differential geometry planted by Riemann few years before. After graduation, Ricci obtained a fellowship that allowed him to spend some years in Munich, in Germany. There he came in touch with the new conception and classification of geometries, based on symmetry groups, developed by Felix Klein and magisterially summarized by him in the celebrated *Erlangen Programme*. These ideas had an analogous strong impact on Luigi Bianchi. Promoted to the position of full-professor at the University of Padova in 1880, Ricci had there an exceptionally talented graduate student: Tullio Levi Civita who was born in that city in 1873. Ricci, Bianchi and Levi-Civita constructed the mathematical language used by Einstein to formulate General Relativity, which is also the most common language for classical differential geometry. The key ingredients of that language are just the tensors whose defining property is that of transforming from one coordinate patch to another, with suitable products of the jacobian matrix of the coordinate transformation:

Felix Klein (1849 -1925)

Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,

o. ö. Professor der Mathematik an der Universität Erlangen.

Programm

zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexanders-Universität zu Erlangen.

Erlangen.

Verlag von Andreas Deichert.

1872.

$$\tilde{t}^{\mu_1\ldots\mu_m}_{\lambda_1\ldots\lambda_n}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial \tilde{x}^{\mu_m}}{\partial x^{\rho_m}} \frac{\partial x^{\sigma_1}}{\partial \tilde{x}^{\sigma_1}} \cdots \frac{\partial x^{\sigma_n}}{\partial \tilde{x}^{\sigma_n}} t^{\rho_1\ldots\rho_m}_{\sigma_1\ldots\sigma_n}(x)$$

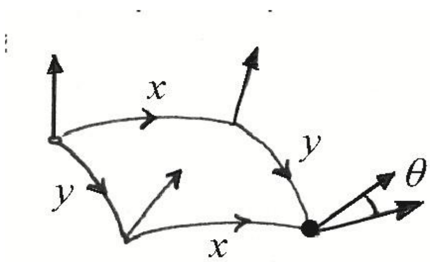
Hence the absolute differential calculus of Ricci, Levi Civita and Bianchi is just the differential calculus for sections of those fiber bundles whose transition functions are completely determined by the very manifold structure of their base-manifold. The concept of covariant differentiation

$$\nabla_\mu t_{\lambda_1\ldots\lambda_n} \equiv \partial_\mu t_{\lambda_1\ldots\lambda_n} - \left\{ \begin{matrix} \rho \\ \mu\,\lambda_1 \end{matrix} \right\} t_{\rho\,\lambda_2\ldots\lambda_n} - \left\{ \begin{matrix} \rho \\ \mu\,\lambda_2 \end{matrix} \right\} t_{\lambda_1\rho\ldots\lambda_n} \cdots - \left\{ \begin{matrix} \rho \\ \mu\,\lambda_n \end{matrix} \right\} t_{\lambda_1\ldots\lambda_{n-1}\rho}$$

was formally developed by Ricci and Levi Civita and, as already stressed above, by using the Christoffel symbols, it realizes the idea of parallel transport preserving the angles defined by a metric structure. Once the covariant differentiation is given, one can consider its antisymmetric square and this leads to the Riemann-Christoffel curvature tensor which, sketched by Riemann in the *Ueber die Hypothesen* and analytically defined by Christoffel, realizes for an arbitrary manifold the idea of intrinsic curvature devised by Gauss in the 1828 *Disquisitiones* :

$$\left[\nabla_\mu , \nabla_\nu\right] V^\rho = V^\sigma R_{\mu\nu\sigma}{}^\rho$$

The geometrical meaning of this relation is exemplified in a simple figure. Consider an infinitesimally small rectangle whose two sides are given by the two vectors X and Y (also of infinitesimally short length), departing from a given point p. Consider next the parallel transport of a third vector V to the opposite site of the rectangle. This parallel transport can be performed along two routes, both arriving at the same destination. The first route follows first X and then Y. The second route does the opposite. The image



Système de Riemann. — Relations entre les éléments du deuxième système dérivé d'un système covariant quelconque.

Soit

$$\varphi = \sum_{1}^{n}{}_{rs}\, a_{rs}\, dx_r\, dx_s$$

la quadrique fondamentale et posons

$$2a_{rs,t} = \frac{\partial a_{rt}}{\partial x_s} + \frac{\partial a_{st}}{\partial x_r} - \frac{\partial a_{rs}}{\partial x_t},$$

$$a_{rs,tu} = \frac{\partial a_{rt,s}}{\partial x_u} - \frac{\partial a_{ru,s}}{\partial x_t} + \sum_{1}^{n}{}_{pq}\, a^{(pq)}(a_{ru,p}\, a_{st,q} - a_{rt,p}\, a_{su,q}).$$

Les symboles $a_{rs,tu}$ sont les éléments d'un système quadruple covariant, qui a une grande importance dans la théorie des quadriques de différentielles. On les trouve dans la *Commentatio mathematica* de Riemann*) (à un facteur numérique près) et c'est à cause de cela que nous désignerons ce système par le nom de *système covariant de Riemann.* — Les expressions $a_{rs,tu}$ furent rencontrées avant la publication du Mémoire cité du grand géométre par M. Christoffel**), qui en mit en évidence les propriétés fondamentales. Il suffira ici de rappeler que le nombre de ces expressions, qui ne sont liées entre elles par aucune relation linéaire, est $N = n^2(n^2 - 1):12$.

vectors of these two transports are based at the same point, so they can be compared. The rotation of one with respect to the other is encoded in the Curvature Tensor, while the translation of one with respect to the other is encoded in the Torsion tensor. In their 1899 paper Ricci e Levi Civita named *Système covariant de Riemann* what now we call Riemann curvature tensor. In the case of the Christoffel symbols the torsion is identically zero, yet for more general connections it can be different from zero and Levi-Civita correctly singled out the vanishing of the torsion as one of the two axioms from which the metric connection can be derived.

In a paper of 1903 Gregorio Ricci introduced a new tensor, later named after him, which is obtained from the Riemann-Christoffel tensor through a contraction of indices. The Ricci tensor is defined as follows:

$$\mathrm{Ric}_{\mu\nu} \equiv \sum_{\rho=1}^{N} R_{\mu\nu\rho}{}^{\rho}$$

and, on a metric manifold, measures the first deviation of its volume form from the euclidian value. Just for this reason it was originally considered by its inventor. Yet such tensor was doomed to play a major role in the development of XXth century scientific thought and in the birth of General Relativity.

Bianchi Identities (1902)

Preparatory to this great future of the Ricci tensor were the algebraic and differential identifies it satisfies. They were derived by Luigi Bianchi in 1902. Actually, according to Levi Civita, the same identities had already been discovered by Ricci as early as 1880 but they had been discarded by their author as not relevant. The first of Bianchi identities states that the Ricci tensor is symmetric:

$$\mathrm{Ric}_{\mu\nu} = \mathrm{Ric}_{\nu\mu}$$

the second, differential identity, states that its divergence is equal to one half of the gradient of its trace:

$$\nabla^{\mu}\, \mathrm{Ric}_{\mu\nu} = \tfrac{1}{2}\nabla_{\nu}\mathrm{R}$$

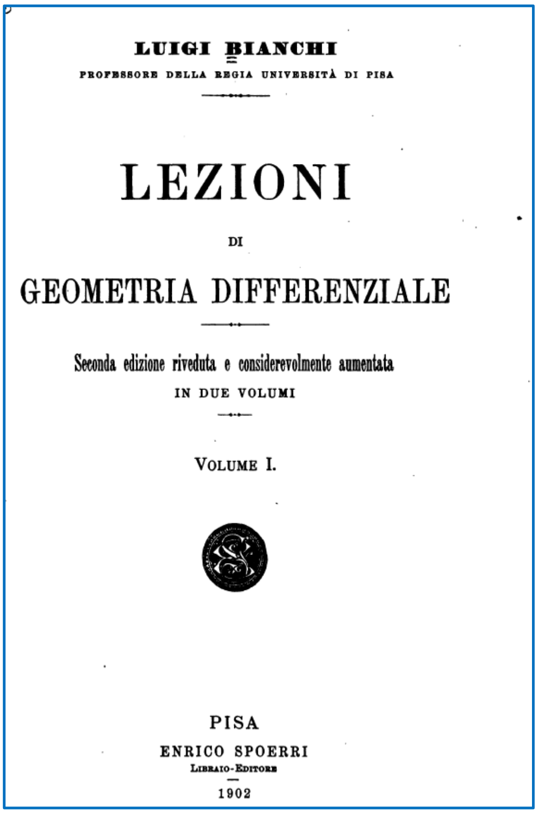
where, by definition, we have posed:

$$\mathrm{R} = g^{\mu\nu}\,\mathrm{Ric}_{\mu\nu}$$

Luigi Bianchi (1865 - 1928)

which is named the curvature scalar. The Bianchi identities were precisely the clue that lead Einstein, with the help of Marcel Grossman, to single out the form of the field equations of General Relativity. Combined in a proper way, they suggest the form of a covariantly conserved tensor, the Einstein tensor, which plays the role of left hand side in the propagation equations, the right hand side being already decided on physical grounds, namely the conserved stress energy-tensor.

After his laurea in Mathematics from the University of Pisa, which he obtained in 1877, Bianchi remained in that city for other two years as student of the *Corso di Perfezionamento* of the *Scuola Normale Superiore*. He graduated in 1879, defending a thesis on helicoidal surfaces. Then, just following the steps of Ricci, he was in Germany, first in Munich and then in Göttingen, were he attended courses and seminars given by Felix Klein. As already stated, he was deeply influenced by Klein's group-theoretical view of geometry and one of his major achievements is precisely along that line. In a paper of 1898, Bianchi classified all tridimensional spaces that admit a continuous group of motions. Actually, so doing, he classified all Lie algebras of dimension three. This classification, which is organized into nine types, turned out to be quite relevant for Cosmology in the framework of General Relativity, since it amounts to a classification of all possible space-times that are spatially homogeneous. Since 1882, Bianchi was internal professor at the Scuola Normale and in 1886 he won the competition for the chair of *Projective Geometry* at University of Pisa, where he was full-professor for the rest of his life. The same year he published the first edition of his *Lezioni di Geometria Differenziale*, which is the very first comprehensive treaty on the new discipline pioneered by Riemann and also the first place where the name Differential Geometry appeared. The second edition quite enlarged and restructured was published in 1902 and contains the famous identities.


LUIGI BIANCHI
PROFESSORE DELLA REGIA UNIVERSITÀ DI PISA

LEZIONI
DI
GEOMETRIA DIFFERENZIALE

Seconda edizione riveduta e considerevolmente aumentata
IN DUE VOLUMI

VOLUME I.

PISA
ENRICO SPOERRI
LIBRAIO-EDITORE
1902

The story we have so far reported reveals the close links between Göttingen and Pisa that stand behind the birth of Differential Geometry. This is just one of the red threads that cross the whole mathematical history of the XIXth century providing the cultural background of General Relativity and of that new vision of the universe, which the plaque posted on Ricci's house advocates. Another no less relevant red thread links Paris, Christiania (now Oslo), Göttingen and Leipzig. The tale associated with such a thread is that of Group Theory and has both romantic and tragic touches. We briefly pause to trace back such thread from its beginning up to the point where it intersects the thread of Differential Geometry.

The Tale of Lie Group Theory

The notion of a group G was invented by *Evariste Galois* in the context of his theory of solubility of algebraic equations by means of radicals. As it is well known, this romantic and very unlucky mathematical genius died in a duel at the age of 22 in 1832. Twice he had tried to publish his spectacular mathematical results and twice he did not succeed for incredibly strange reasons. The first time his referee lost the manuscript before reading it, the second time the referee died the very same night he received the paper for reviewing and no one among his heirs paid attention to those incomprehensible pages. In the last two years of his life Evariste was twice arrested as a subversive, spent some months in prison, was released, participated to other political quarrels, had a love affair with a girl of vulgar personality, who disgusted him also on that

Evariste Galois (1810 - 1832)

front, finally was involved in a stupid debate with a political exponent of opposite views, that ended up in the duel which caused his death. Perfectly aware of being confronted with almost sure death, the night before the duel, Evariste wrote an exposition of all his mathematical results that he gave to his loyal friend Auguste Chevalier. Fortunately, this latter did not loose the sixty pages received from Galois and in 1846 Galois main theorem was finally published on the *Journal de Mathématiques Pures et Appliquées*, with the praising comments of its main editor, namely *Joseph Liouville*. Once the notion of a transformation group G is introduced, the notion of equivalence classes naturally arises. A set of objects acted on by G can be rationalized by dividing it into stocks, each of which contains all those that are mapped one into the other by some transformation of the group. In some sense all the objects that happen to be in the same stock are different realizations of the same entity which is none of them, but just the entire equivalence class.

Directly influenced by Galois' ideas that came to them through *Darboux* and *Jordan*, *Sophus Lie* and *Felix Klein* started rethinking classical geometry from a new viewpoint. In particular Klein realized that Euclid axiomatic definitions of what is an equilateral triangle, a rectangular triangle and so on, can be recast into the notion of equivalence classes. There are many triangles that one can draw in a plane but two triangles that can be mapped one into other by means of a rotation or a translation, namely an element of what we name the Euclidian Group E_2 have to be identified and considered just the same triangle. Hence the objects of study in Euclidian Geometry are just the equivalence classes with respect to E_2. It follows immediately that all the propositions of Euclidian Geometry are just statements on properties and relations that are invariant with respect to E_2 or in three-space with respect to E_3. In this way Klein came to conceive the momentous Erlangen Programme. Since there are other groups different from the Euclidian Group, you can conceive other geometries, among which the non-Euclidian one introduced by Lobachevsky. Actually you can classify geometries according to the group G with respect to which the relations considered in that geometry are invariant.

Sophus Lie (1842 - 1899)

In 1870 the *Collegium Academicum* in Christiania gave to the young Norwegian mathematician *Sophus Lie*, fascinated by Plücker's conceptions of Geometry a research–travel grant that allowed him to go to Berlin, Göttingen and eventually to Paris. in Berlin, Sophus Lie met with Felix Klein who had studied in Bonn precisely under the supervision of Plücker, passed away two years before. The two young scientists had a lot of interests in common and became immediately close friends, although, as Freudenthal remarks, they had quite different characters both as humans and as mathematicians. They traveled together to Paris where they met and interacted with Gaston Darboux and Camille Jordan.The conversations with Jordan were of the highest relevance for both Lie and Klein since the French mathematician attracted their attention to the role that group–theory could play in geometry. For Lie this was the germ of a reasoning that conducted him to the notion of transformation groups. Klein developed these ideas in what two years later appeared as the Erlangen Programme. In any case Lie and Klein discussed intensively about these issues and eventually published a common work. They lived in adjoining rooms in the same hotel and saw each other continuously. Few days after these scientific events, Napoleon the third, falling into Bismarck's trap, declared war to Prussia and hostilities began (July 19th 1870). Being a citizen of Prussia, Klein had to flee immediately from France, while Lie, who was a citizen of Norway, namely of a neutral state, remained. In August, when the Prussians had already trapped part of the French Army in Metz, Lie decided to leave Paris and hike towards Italy. When he reached Fontainebleau he was arrested as a German spy and his mathematical notes, written in German, were used as an evidence against him, regarding them as ciphered messages. He spent several weeks in prison and was finally released thanks to the intervention of Darboux who explained the case to the suspicious police. Once he was freed, Lie fled to Italy and from there he made his way back to Norway through Germany.

In 1871, back in Christiania, Lie completed his PhD doctoral thesis on the basis of his Paris discoveries and he was awarded his doctorate in 1872. The same year the University of Christiania created a new chair on which he was appointed.

In 1872, at the age of 23, Felix Klein was appointed Full Professor at the University of Erlangen, where he remained only three years, since in 1875 he received and accepted an offer from the Technische Hochschule of Münich. There he remained longer, namely five years, and accomplished important steps both in his personal and professional life. As for personal life, Münich was the city where, in August 1875, he married with Anne Hegel, the granddaughter of the philosopher Georg Wilhelm Friedrich Hegel. On the scientific side, Klein worked very much intensively in Münich and his fame as a brilliant and profound teacher spread through the world attracting there students that later became famous mathematicians and physicists among them Max Planck, Adolf Hurwitz and Ricci Curbastro.

Once appointed to professorship in Christiania in 1872, Lie started working on partial differential equations. He wrote: "*the theory of differential equations is the most important discipline in modern mathematics*." The influence of their group discussions in Paris motivated Lie in a direction different from the geometrical one pursued by Klein in Bavaria. After the interactions with Jordan he was under the strong impression of Galois theory about which he had previously heard from Sylow in his student years, without paying too much attention. He wanted to uplift to the level of differential equations what Galois had done for the algebraic ones. In a paper of 1874 he wrote: "*How can knowledge of a stability group for a differential equation be utilized towards its integration?"* By stability group of a differential equation it was meant a group of transformations whose effect was that of permuting the solutions of the equations among themselves. Pondering on such questions Lie came to develop the theory of continuous groups of transformations and making them infinitesimal he arrived at the notion of Lie algebra satisfied by the vector fields Ψ_α that generate such transformations:

$$\left[\boldsymbol{\Psi}_\alpha , \boldsymbol{\Psi}_\beta\right] = \sum_{\gamma=1}^{r} c_{\alpha\beta}^{\ \ \gamma} \boldsymbol{\Psi}_\gamma$$

and stated the so named Jacobi identities satisfied by the structure constants $c_{\alpha\beta}{}^{\gamma}$:

$$c_{\alpha\beta}^{\ \ \gamma} = -c_{\beta\alpha}^{\ \ \gamma}$$

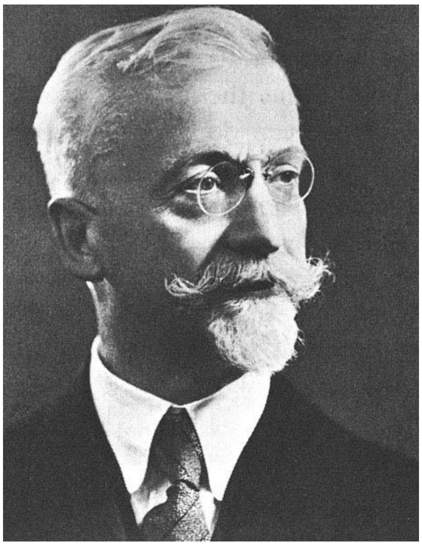
$$0 = \sum_{\sigma=1}^{r} \left(c_{\alpha\beta}^{\ \ \sigma} c_{\sigma\delta}^{\ \ \gamma} + c_{\beta\delta}^{\ \ \sigma} c_{\sigma\alpha}^{\ \ \gamma} + c_{\delta\alpha}^{\ \ \sigma} c_{\sigma\beta}^{\ \ \gamma} \right)$$

In the following years the lives of Klein and Lie intersected each other in many ways. In 1876 Klein left Münich for Leipzig, whose University offered him a prestigious Chair of Geometry. He had not forgotten his good friend Lie and knowing about his isolation in Norway, Klein organized to send him his own student Friedrich Engel who helped Lie in the course of nine years. In 1886 Felix Klein changed once again his location accepting the offer of Göttingen University, whose world leadership in Mathematics and Physics Klein strongly helped to further strengthen, in particular with the appointment of David Hilbert. The vacant Chair of Geometry in Leipzig was immediately offered to Lie, who accepted and lived in Germany for twelve years up to 1898. In 1886, the same year he had joined the Faculty in Leipzig, Lie received the visit of an obscure school principal arriving from distant Easter Prussia. This was Wilhelm Killing who, two years before, in 1884, had sent to Klein a small booklet, printed in his school and humbly named *Programmschrift,* where *Lie algebras*, discovered by him independently from Lie, were presented under a different name, together with the notion of *simple Lie algebras,* which the same Lie never considered. Following Klein's suggestions, Killing had corresponded by mail with both Lie and Engel and now, using the opportunity of a work-trip to Heidelberg, he had come to Leipzig in order to show Lie his results. The bad-tempered Lie, always very jealous of his own results and obsessed with the idea of getting insufficient recognition for his own work, was ill–disposed towards this humble school teacher, coming from nowhere in the far east and claiming to have independently obtained Lie algebras. The meeting was a complete failure and Killing continued his journey, remaining however on good terms with Engel. October 18th 1887, Killing wrote to Engel announcing that he had found the complete list of simple Lie Algebras, any semisimple one being a tensor sum of the latter. Indeed Killing had already invented the formalism of roots and he had constructed the complete classification of simple Lie algebras, including the exceptional ones G_2,F_4,E_6,E_7,E_8 . All of Killing's results were published between 1888 and 1889 on the prestigious journal Mathematische Annalen founded by Klein. For the rest of his life-time, that extended until 1923, Killing was absorbed by teaching, administration and charitable work. It was in 1894 the turn of Cartan to continue to marvelous tale of Lie algebras. Cartan's doctoral dissertation was presented in that year and was already a masterpiece. His thesis was a rigorous remake of Killing's papers where he also gave the explicit matrix construction of all exceptional Lie algebras, already announced in a paper published by him one year before in German. Of very humble origin, being the son of a poor blacksmith in the mountain village of Dolomieu in Haute Savoie, Cartan obtained the very best scientific education available at the time thanks to the state-stipends that the French Republic had introduced for talented people, independently from their social or economical status. Discovered in his remote village by the school inspector Dubost, Elie was state-supported in order to attend Lycée in Lyon and then entered the Ecole Normale Superiéure of Paris where he had such masters as Picard, Darboux and Hermite, becoming one of the most prominent mathematician of the XXth century and probably of all times.

Wilhelm Karl Joseph Killing (1847-1923).

The two threads joined

Thus we see how the two red threads of Curved Geometry and of Group Theory intersected and brought the mathematical language and the weaponry needed by the new Physics of the XXth century to maturity. In 1858 Enrico Betti, professor at the University of Pisa, visited Göttingen, Berlin and Paris, making many important mathematical contacts. In Göttingen Betti met Riemann and developed friendship with him. In an attempt to improve his health, Riemann made an Italian visit in the autumn of 1863 and renewed his



Elie Cartan (1869 -1951)


Tullio Levi-Civita (1873-1941)

friendship with Betti. Ricci Curbastro and Luigi Bianchi were both students of Betti and learnt from him about Riemann ideas on curved manifolds. During their stays in Germany, respectively in Munich and in Gottingen, they absorbed Klein's views on the role of groups in geometry and certainly they became early acquainted with Lie's work on Lie algebras. The French tradition, going back to Galois, on whose theory Betti himself had worked extensively, got mixed with the German tradition originating from Gauss' work on curved surfaces and brought the Italian Masters of the New Differential Geometry to that top at world-level that allowed them to pave Einstein's path toward General Relativity.

Bianchi, Ricci and Levi Civita: from 1902 to 1941

Bianchi died in 1928 and he is buried in the Cimitero Monumentale, Piazza dei Miracoli of Pisa. Since the later 1880.s up to the end of his life he was an extremely prominent and influential mathematician of the then flourishing Italian School of Geometry. In 1904 Bianchi was member of the committee appointed by the Academia Nazionale dei Lincei to select the winning paper for the Royal Prize of Mathematics. Ricci's ambitions on that Prize had already been manifested some years before, when he presented his works to the committee then headed by Eugenio Beltrami. Notwithstanding Beltrami's very favorable impressions, the final verdict of the jury on the relevance of tensor analysis had been hesitating and the Prize had not been attributed. Similar conclusion obtained the competition of 1904. Luigi Bianchi showed a great appreciation for the mathematical soundness and vastity of Ricci's methods but concluded that tensor analysis had not yet demonstrated its relevance and essentiality. He utilized Kronecker's words to say that *he preferred new results found with old methods rather than old results retrieved with new, although very powerful, techniques*. This sentence can be compared with the Poincaré sentence reported by Levi-Civita and Ricci at the beginning of their 1899 paper. These events are moreover surprising in view of the fact that two years before, in 1902, Bianchi had published his paper containing those identities on the Ricci tensor for which his name is mostly remembered. The Royal Prize for Mathematics, denied to Ricci Curbastro, was attributed few years later, in the 1907 edition, to Ricci's former student Tullio Levi Civita, by a committee that once again included Luigi Bianchi, together with other distinguished mathematicians such as Vito Volterra and Corrado Segre. This time the usefulness of the tensor methods had been made absolutely undoubtable by the vastity of Levi-Civita's results. Although a little bit dismayed by the failure to get the Royal Prize, Ricci Curbastro ended his life in 1925 surrounded by the appreciation of his colleagues and of his fellow citizens both as a scientist and as a politician. Indeed he was nominated member of several academies, including the most prestigious one, that of Lincei and also occupied positions in the local administration of his native city, Lugo di Romagna. On the contrary his genial student Levi-Civita, who was professor at the University of Rome La Sapienza, notwithstanding the Royal Prize and other honors, suffered, under the fascist racial laws of 1938, the removal from his chair because of his Jewish origin. Depressed and completely isolated from the scientific world he died from sorrow in 1941. It is a luminous shot in a dark and barbarous time that when he was removed from his Chair at la Sapienza, Levi-Civita was offered a chair by the *Academia Pontificia*.