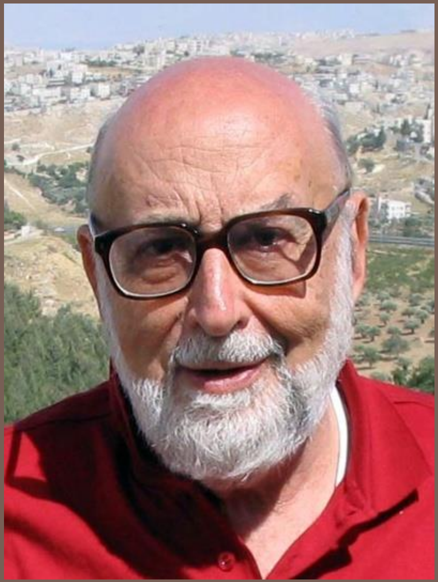

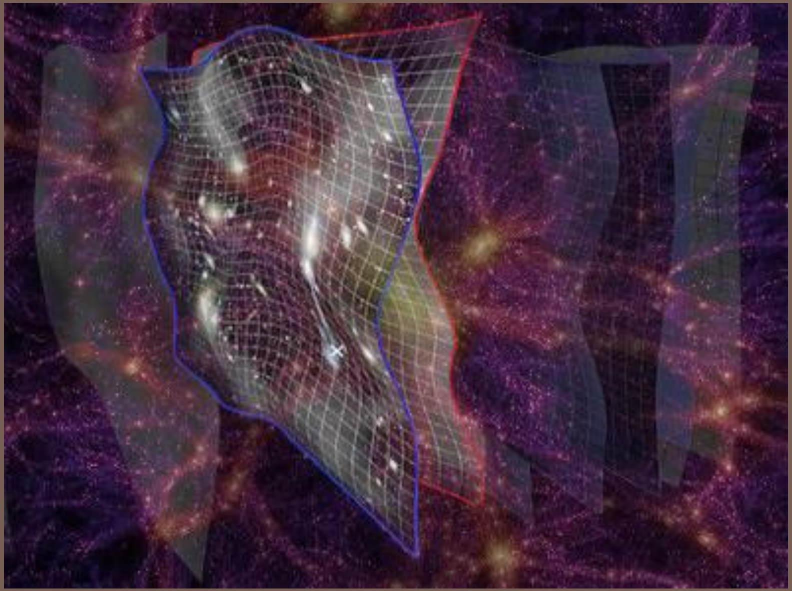

M2-BRANES WITH ENGLERT FLUXES AND THE GROUP PSL(2,7)

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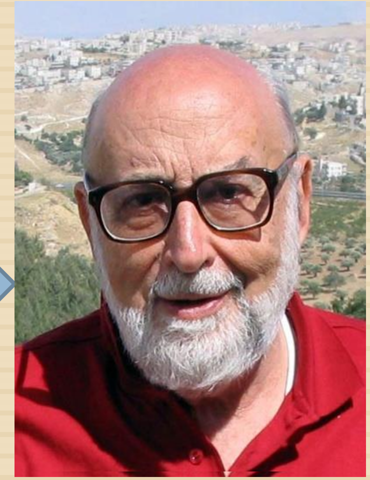
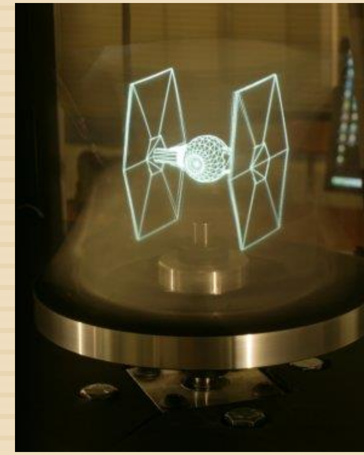
Torino University & Embassy of Italy in the Russian Federation

Lebedev Institute January 22nd 2016

For those who heard me before

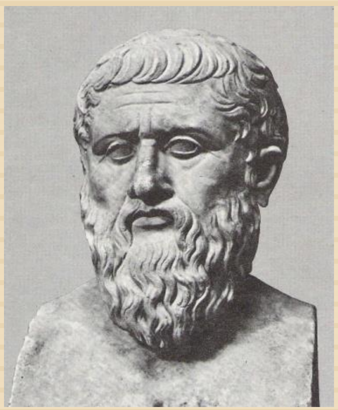
Uplifting





M2-branes

Gauge Theories in D=3 with discrete symmetry Γ



Uplifting

This tale starts from the filling of a slot left empty in my December 2015 talk in Dubna…….

$$\Gamma = \mathrm{PSL}(2, \mathbb{Z}_7)$$

Felix Klein 1879

Pietro Fre' arXiv:1601.02253 [hep-th]

D=11 Supergravity

$$
\begin{aligned}
\mathfrak{T}^a &= \mathcal{D}V^a - \mathrm{i}\frac{1}{2}\overline{\psi}\wedge\Gamma^a\psi \\
\mathfrak{R}^{ab} &= d\omega^{ab} - \omega^{ac}\wedge\omega^{cb} \\
\rho &= \mathcal{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab}\wedge\Gamma_{ab}\psi \\
\mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} - \frac{1}{2}\overline{\psi}\wedge\Gamma_{ab}\psi\wedge V^a\wedge V^b \\
\mathbf{F}^{[7]} &= d\mathbf{A}^{[6]} - 15\,\mathbf{F}^{[4]}\wedge\mathbf{A}^{[3]} - \frac{15}{2}V^a\wedge V^b\wedge\bar{\psi}\wedge\Gamma_{ab}\psi\wedge\mathbf{A}^{[3]} \\
&\quad -\mathrm{i}\frac{1}{2}\overline{\psi}\wedge\Gamma_{a_1\dots a_5}\psi\wedge V^{a_1}\wedge\dots\wedge V^{a_5}
\end{aligned}
$$

The Free Differential Algebra

$$
\begin{aligned}
\mathfrak{T}^a &= 0 \\
\mathbf{F}^{[4]} &= F_{a_1\dots a_4}V^{a_1}\wedge\dots\wedge V^{a_4} \\
\mathbf{F}^{[7]} &= \frac{1}{84}F^{a_1\dots a_4}V^{b_1}\wedge\dots\wedge V^{b_7}\epsilon_{a_1\dots a_4 b_1\dots b_7} \\
\rho &= \rho_{a_1a_2}V^{a_1}\wedge V^{a_2} - \mathrm{i}\frac{1}{3}\left(\Gamma^{a_1a_2a_3}\psi\wedge V^{a_4} + \frac{1}{8}\Gamma^{a_1\dots a_4 m}\psi\wedge V^m\right)F^{a_1\dots a_4} \\
\mathfrak{R}^{ab} &= R^{ab}{}_{cd}V^c\wedge V^d + \mathrm{i}\,\rho_{mn}\left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a}\delta^{b]c} + 2\,\Gamma^{ab[m}\delta^{n]c}\right)\psi\wedge V^c \\
&\quad +\overline{\psi}\wedge\Gamma^{mn}\psi\,F^{mnab} + \frac{1}{24}\overline{\psi}\wedge\Gamma^{abc_1\dots c_4}\psi\,F^{c_1\dots c_4}
\end{aligned}
$$

The rheonomic parameterization

The field equations of M-theory

$$
\begin{aligned}
0 &= \mathcal{D}_m F^{mc_1c_2c_3} + \frac{1}{96}\,\epsilon^{c_1c_2c_3a_1a_8}\,F_{a_1\dots a_4}\,F_{a_5\dots a_8} \\
0 &= \Gamma^{abc}\,\rho_{bc} \\
R^{am}{}_{cm} &= 6\,F^{ac_1c_2c_3}\,F^{bc_1c_2c_3} - \frac{1}{2}\,\delta^a_b\,F^{c_1\dots c_4}\,F^{c_1\dots c_4}
\end{aligned}
$$

These are very simple equations just as much elegant as Einstein equations themselves, with an incredible wealth of solutions . Their deep meaning so far no-one has entirely appreciated. During the last 35 years these equations have continued to reveal new unexpected aspects and new profound and surprising connections.

M2-brane solutions

LOCAL TOPOLOGY OF SPACE-TIME

$$\mathcal{M}_{11} = \mathrm{Mink}_{1,2} \times \mathbb{R}_+ \times \mathrm{T}^7$$

BRANE METRIC

$$ds^2_{11} = H(y)^{-\frac{2}{3}} \left(d\xi^\mu \otimes d\xi^\nu \, \eta_{\mu\nu}\right) - H(y)^{\frac{1}{3}} \left(dy^I \otimes dy^J \, \delta_{IJ}\right)$$

$$y^0 = U \in \mathbb{R}_+ \quad ; \quad y^i = x^i \in \mathrm{T}^7 \quad (i = 1, \dots, 7)$$

THE FLUXES

$$\mathbf{A}^{[3]} = \frac{2}{H(y)} \Omega^{[3]} + e^{-\mu U} \mathbf{Y}^{[3]}$$

$$\Omega^{[3]} = \frac{1}{6} \epsilon_{\mu\nu\rho} \, d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho$$

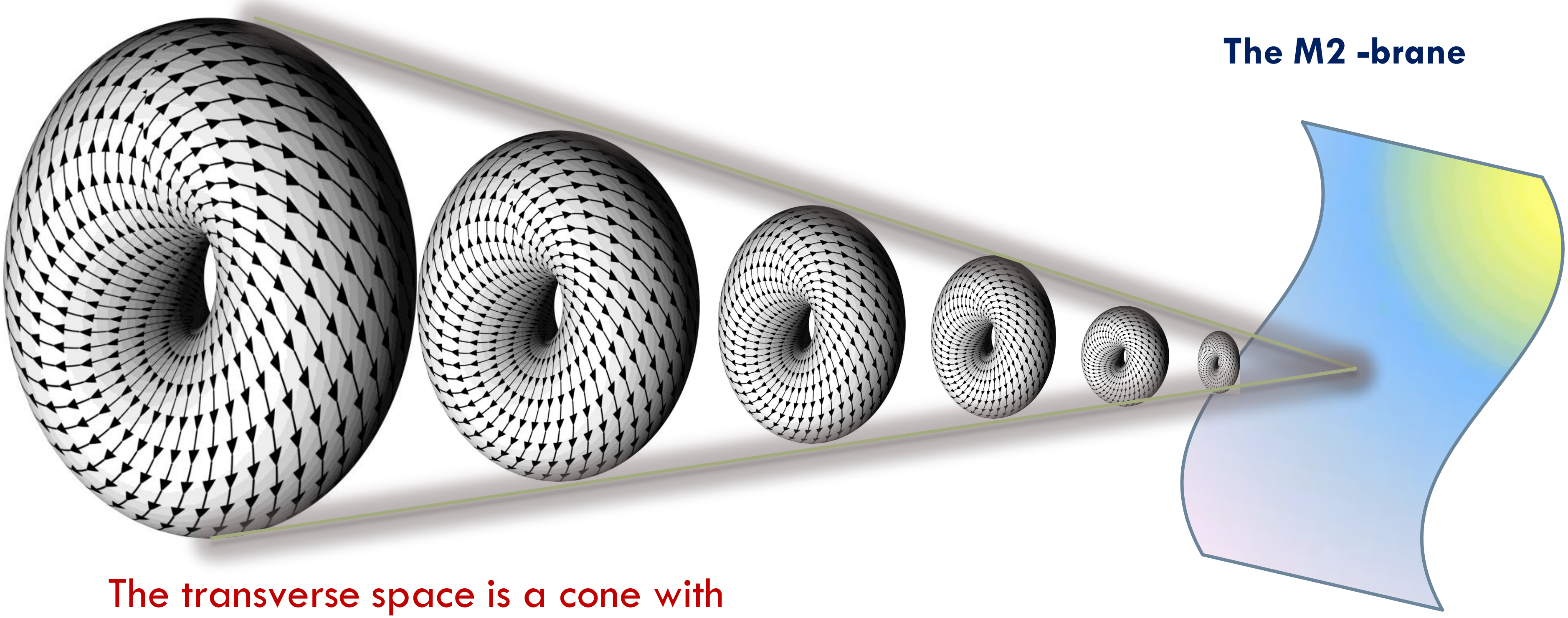
ENGLERT EQUATION

$$\star_{\mathrm{T}^7} \mathrm{d}\mathbf{Y}^{[3]} = -\frac{\mu}{4} \mathbf{Y}^{[3]}$$

HARMONIC EQUATION

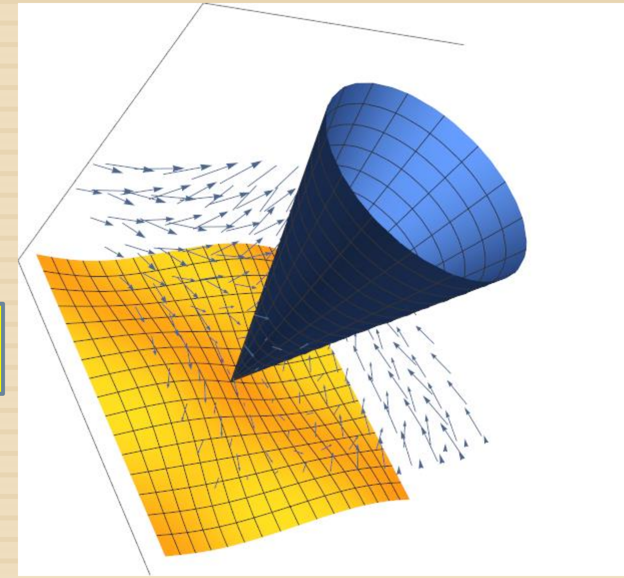
$$\Box_{\mathbb{R}_+ \times \mathrm{T}^7} H(y) = -\frac{3\mu^2}{2} e^{-2\mu U} \parallel \mathbf{Y} \parallel^2 \equiv J(y)$$

Global Structure

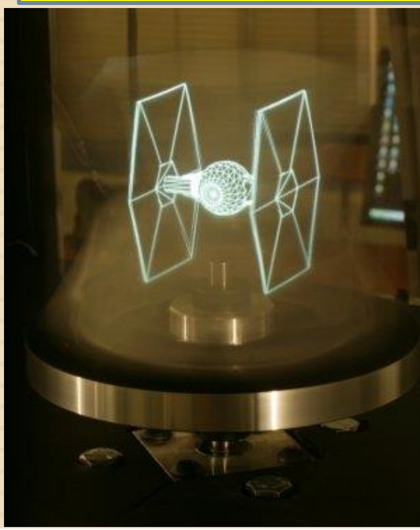


The transverse space is a cone with base T^7

The goal is eventually the holographic principle. The 3D-gauge theories on the brane world-sheet will inherit the discrete symmetries of the Englert fluxes in the transverse space that produce the supergravity solution



$$\Gamma = \text{discrete symmetry of } \mathbf{Y}^{[3]}$$



$$\star\, \mathrm{d}\mathbf{Y}^{[3]} = -\frac{\mu}{4}\,\mathbf{Y}^{[3]}$$

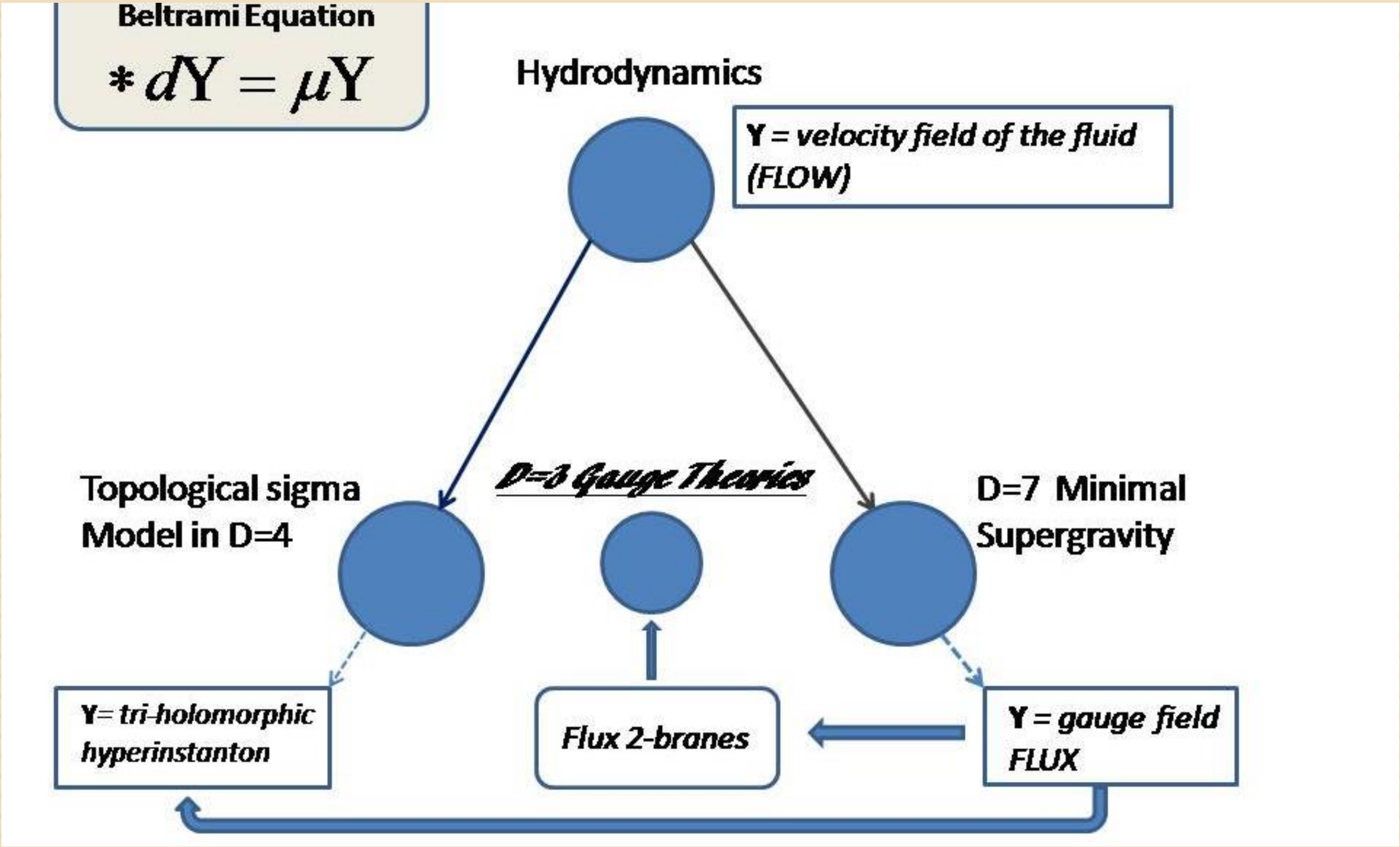
The holographic principle

ENGLERT EQUATION has an ANCESTOR

Englert Equation is the generalization to 3-forms in d=7 of BELTRAMI EQUATION holding on 1-forms in d=3

$$\star\, \mathrm{d}\mathbf{Y}^{[1]} = -\,\mu\,\mathbf{Y}^{[1]}$$

Let us review what we discovered about M2 branes with Arnold Beltrami fluxes in 2015



A research program developed in 2015

- P. Fre , A.S. Sorin arXiv:1501.04604, arXiv:1504.06802
- P. Fre, P. A. Grassi, A.S. Sorin arXiv:1509.09056
- P.Fre, P. A. Grassi, L. Ravera, M. Trigiante arXiv:1511.06245

Euler equation of classical hydrodynamics

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \quad ; \quad \nabla\cdot\mathbf{u} = 0$$

Mathematical Hydrodynamics

$$\mathcal{S} : \mathbb{R}_t \to \mathcal{M}_g$$

A flow is a smooth map from the time line to a Riemannian manifold

$$\forall t \in \mathbb{R} : u^i(x,t)\,\partial_i \equiv \mathsf{U}(t) \in \Gamma(T\mathcal{M}, \mathcal{M})$$

The velocity field is a section of the tangent bundle

$$\forall t \in \mathbb{R} : \Omega^{[\mathsf{U}]}(t) \equiv g_{ij}\,u^i(x,t)\,dx^j \in \Gamma(T\mathcal{M}, \mathcal{M})$$

lowering the indices we have a 1-form

Rewriting of Euler equations we obtain

$$-\mathrm{d}H = \partial_t\Omega^{[\mathsf{U}]} + \mathrm{i}_{\mathsf{U}}\cdot\mathrm{d}\Omega^{[\mathsf{U}]}$$

$$H = \left(p + \tfrac{1}{2}\parallel \mathsf{U} \parallel^2\right)$$

If H depends on x $\in$ M, the streamlines occur on level surfaces H=const and are two-dimensional. A necessary condition for chaos is

$$0 = \mathrm{i}_{\mathsf{U}}\cdot\mathrm{d}\Omega^{[\mathsf{U}]}$$

On a 3-torus

$$T^3 = \frac{\mathbb{R}^3}{\Lambda}$$

$$\star\, \mathrm{d}\mathbf{Y} = -\mu\, \mathbf{Y}$$

Beltrami Equation

Arnold Theorem

There are only two possibilities

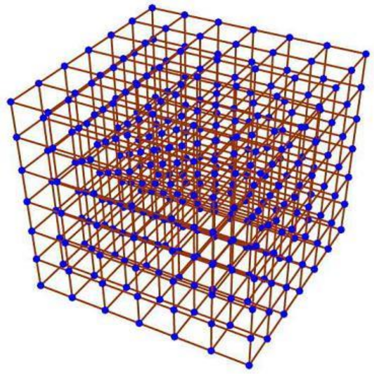
a) Either the form $\Omega^{[U]}$ is an eigenstate of the Beltrami operator $\star_g \mathrm{d}$ with a non vanishing eigenvalue $\lambda \neq 0$

$$\star_g\, \mathrm{d}\Omega^{[U]} = \lambda\, \Omega^{[U]}$$

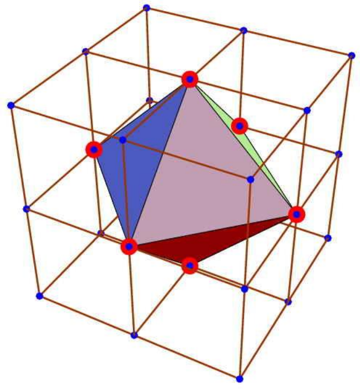
b) or the manifold $\mathcal{M}$ is subdivided into a finite collection of cells, each of which admits a foliation diffeomorphic to $T^2 \times \mathbb{R}$ and every two-torus T^2 is an invariant set with respect to the action of the velocity field U: in other words, all trajectories lay on some T^2 immersed in the manifold $\mathcal{M}$.

Two-dimensional streamlines = no chaos

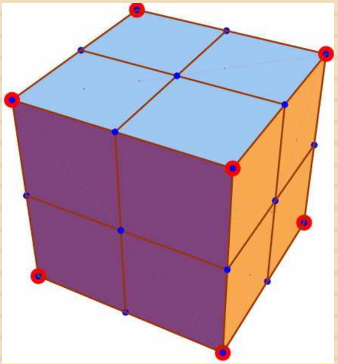
The cubic lattice is self –dual. The rotation subgroup of invariance is the octahedral group O_{24}



The cubic lattice

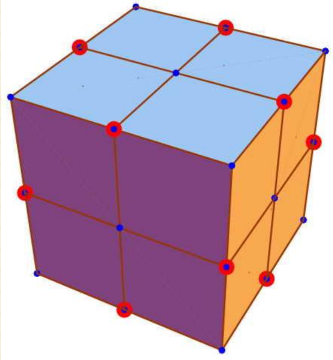


The orbit of legth 6

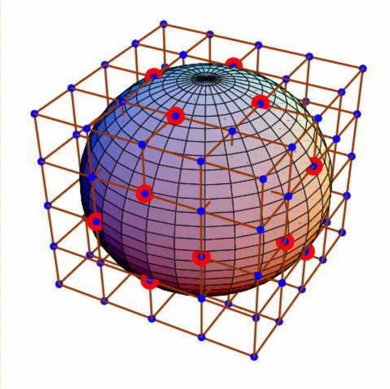


The orbit of legth 8

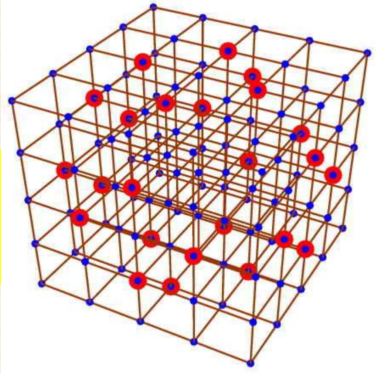
In the momentum lattice there are 4 types of orbits under O_{24}



The orbit of legth 12



The orbit of length 24



Classification of Beltrami fields on T^3

$Y\,(X|F)$

Obtained by Fre & Sorin in 2014-2015

To each O_{24}-orbit in momentum lattice of length r we associate a solution of Beltrami equation depending on r parameters **F** and in one case on 2r parameters **F**

From T^3 to T^7

$$\forall \gamma \in \mathrm{O}_{24} : \mathbf{Y}^{[1]}\left(\gamma \cdot \vec{\mathbf{X}} \,|\, \mathbf{F}\right) = \mathbf{Y}^{[1]}\left(\vec{\mathbf{X}} \,|\, \mathfrak{R}[\gamma] \cdot \mathbf{F}\right)$$

octahedral rotations are represented by linear transformations of parameters

Beltrami fields can be classified into irreducible representations of O_{24}

We can use solutions of Beltrami Equation to produce solutions of Englert Equation

$$\mathrm{T}^7 = \mathrm{T}^3 \otimes \mathrm{T}^4$$

$$\mathbf{x}_{||} = x^{\mathrm{i}} \; (\mathrm{i},\mathrm{j},\mathrm{k} = 1,2,3)$$

$$\mathbf{x}_{\perp} = x^{\alpha} \; (\alpha,\beta,\gamma = 4,5,6,7)$$

$$\mathbf{Y}^{[3]}[\mathbf{W}] = \sum_{\Lambda=1}^{3} \mathbf{W}^{\Lambda} \wedge \mathbb{K}^{\Lambda}$$

$$\mathbb{K}^{\Lambda} = J^{\Lambda|-}_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$$

$$\star_{\mathrm{T}^3}\, \mathrm{d}\mathbf{W}^{\Lambda} = \mu\, \mathbf{W}^{\Lambda}$$

$$\star_{\mathrm{T}^7}\, \mathrm{d}\mathbf{Y}^{[3]}[\mathbf{W}] = -\frac{\mu}{4}\mathbf{Y}^{[3]}[\mathbf{W}]$$

Beltrami

Englert

We need crystallography in d=7…

Before that it is convenient to study supersymmetry

We can formulate the following general question:

How many supersymmetries are preserved by an M2-brane with Englert fluxes?

The problem can be addressed in general terms wit a convenient gamma matrix basis.

Gamma matrices

d=3 $\{\gamma_{\underline{a}}, \gamma_{\underline{b}}\} = 2\eta_{\underline{ab}} \quad ; \quad \gamma = \{\sigma_2, \mathrm{i}\,\sigma_1, \mathrm{i}\,\sigma_3\}$

d=7 $\{\tau_i, \tau_j\} = -2\delta_{ij}$ $\left\{ \begin{array}{lll} (\tau_i)_{jk} & = & \phi_{ijk} \\ (\tau_i)_{j8} & = & \delta_{ij} \quad ; \quad (\tau_i)_{8j} = -\delta_{ij} \end{array} \right.$

octonion structure constants

d=8 $\{T_I, T_J\} = -2\delta_{IJ}$ $\left\{ \begin{array}{lll} T_0 & = & \mathrm{i}\sigma_2 \otimes \mathbf{1}_{8\times 8} \\ T_i & = & \sigma_1 \otimes \tau_i \\ T_9 & = & \sigma_3 \otimes \mathbf{1}_{8\times 8} \end{array} \right.$

d=11 $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$ $\left\{ \begin{array}{lll} \Gamma_{\underline{a}} & = & \gamma_{\underline{a}} \otimes T_9 \\ \Gamma_I & = & \mathbf{1}_{2\times 2} \otimes T_I \end{array} \right.$

Generic spinor

$$\underbrace{\xi}_{32} = \underbrace{\epsilon}_{2} \otimes \underbrace{\kappa}_{2} \otimes \underbrace{\lambda}_{8}$$

Making a long story short

Killing spinor equation

$$\mathcal{D}\xi - \frac{\mathrm{i}}{3}\Gamma^{\mathrm{abc}}V^{d}F_{\mathrm{abcd}}\xi - \frac{\mathrm{i}}{24}\Gamma_{\mathrm{abcdf}}F^{\mathrm{abcd}}V^{f}\xi = 0$$

$$\mathcal{D}\xi \equiv \mathrm{d}\xi - \frac{1}{4}\omega^{\mathrm{ab}}\Gamma_{\mathrm{ab}}\xi$$

Solution

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \qquad \kappa = \begin{pmatrix} H(y)^{-\frac{1}{6}} \\ 0 \end{pmatrix}$$

$$\mathcal{B}\,\lambda = 0$$

$$\mathcal{B} \equiv \tau_{ijk}\,Y_{ijk}$$

The number of preserved supersymmetries , in the d=3 sense, is equal to the number of null eigenvectors of the operator $\mathcal{B}$

# of SUSY charges = 2 x rank $\mathcal{B}$

Englert fluxes built from Beltrami fluxes have too much SUSY…..

We can easily prove that for solutions of Englert Equation coming from Beltrami fields we always have

$$\text{rank}\,\mathcal{B} \geq 4$$

We have to find 7-dimensional torii that do not split in 3+4

$$T^7 \simeq \frac{\mathbb{R}^7}{\Lambda}$$

Λ must be a lattice with a crystallographic Point Group that has elements of maximum order 7 not of maximum order 4. This consideration leads us to a remarkable choice and to a miraculous match

In 1879 Felix Klein studied....

The locus $\mathcal{K}_4 \subset \mathbb{P}^2(\mathbb{C})$ cut by the quartic $x^3 y + y^3 z + z^3 x = 0$

The Klein quartic is a Hurwitz surface because it has genus g=3 and admits the maximal number $84 \times$ (g-1) of conformal automorphisms. The Hurwitz group of this surface was derived by Klein and it is a simple group

$$\mathrm{L}_{168} \equiv \mathrm{PSL}(2,\mathbb{Z}_7)$$

fifteen years ago in a context of nuclear/atomic physics it was proved that:

$$\mathrm{L}_{168} \subset \mathrm{G}_{2(-14)}$$

This gave me the inspiration that L_{168} might be the Point Group of a d=7 lattice Λ from which we might obtain N=1
M2-branes with Englert fluxes. It was just a hint but it came out to be TRUE.

The Hurwitz group L_{168} =PSL(2,7)

Presentation

$$L_{168} = \left(R, S, T \parallel R^2 = S^3 = T^7 = RST = (TSR)^4 = \mathrm{e}\right)$$

Conjugacy Classes

Conjugacy class	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
representative of the class	e	R	S	TRS	T	SR
order of the elements in the class	1	2	3	4	7	7
number of elements in the class	1	21	56	42	24	24

Character Table

Representation	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$
$D_1\,[L_{168}]$	1	1	1	1	1	1
$D_6\,[L_{168}]$	6	2	0	0	-1	-1
$D_7\,[L_{168}]$	7	-1	1	-1	0	0
$D_8\,[L_{168}]$	8	0	-1	0	1	1
$DA_3\,[L_{168}]$	3	-1	0	1	$\frac{1}{2}\left(-1 + \mathrm{i}\sqrt{7}\right)$	$\frac{1}{2}\left(-1 - \mathrm{i}\sqrt{7}\right)$
$DB_3\,[L_{168}]$	3	-1	0	1	$\frac{1}{2}\left(-1 - \mathrm{i}\sqrt{7}\right)$	$\frac{1}{2}\left(-1 + \mathrm{i}\sqrt{7}\right)$

My little discovery

The group L_{168} is crystallographic in d=7. The lattice Λ is the root lattice of A_7

I have found three **integer valued** 7×7 matrices $\mathcal{R},\mathcal{S},\mathcal{T}$ such that

1) $\mathcal{R}^2 = \mathcal{S}^3 = \mathcal{T}^7 = \mathcal{RST} = (\mathcal{TSR})^4 = 1_{7\times 7}$

2) $\mathcal{R}^T \mathcal{C} \mathcal{R} = \mathcal{S}^T \mathcal{C} \mathcal{S} = \mathcal{T}^T \mathcal{C} \mathcal{T} = \mathcal{C}$

$$\mathcal{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Cartan matrix of A_7

For the interested ones…..

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From these generators one reconstructs all the 168 elements of the group and, of course, they are all integer valued and respect the A_7 Cartan matrix.

Obviously, by duality, the group has a crystallographic action also on the dual lattice namely on the A_7 weight lattice.

Where ϕ_{ijk} are the octonionic structure constants transformed to the lattice basis. This proves that L_{168} is a subgroup of compact G_2

$$\begin{aligned} \varphi_{ijk} &= (\mathcal{R})_i^{\ p}\,(\mathcal{R})_j^{\ q}\,(\mathcal{R})_k^{\ r}\,\varphi_{pqr} \\ \varphi_{ijk} &= (\mathcal{S})_i^{\ p}\,(\mathcal{S})_j^{\ q}\,(\mathcal{S})_k^{\ r}\,\varphi_{pqr} \\ \varphi_{ijk} &= (\mathcal{T})_i^{\ p}\,(\mathcal{T})_j^{\ q}\,(\mathcal{T})_k^{\ r}\,\varphi_{pqr} \end{aligned}$$

Solutions of Englert Equation from weight lattice orbits

$$ds^2_{\mathrm{T}^7} = \mathcal{C}_{ij}\, dX^i \otimes dX^j = \delta_{ij}\, dx^i \otimes dx^j \qquad x = \mathfrak{M}\, X$$

Periodicity on the lattice imposes the following expansion

$$\begin{aligned}\mathbf{Y}^{[3]} &= \sum_{\mathbf{k}\in\mathcal{O}} \Big(v_{ijk}(\mathbf{k}) \cos[2\pi\,\mathbf{k}\cdot\mathbf{X}] + \omega_{ijk}(\mathbf{k}) \sin[2\pi\,\mathbf{k}\cdot\mathbf{X}] \Big)\, dX^i \wedge dX^j \wedge dX^k \\ &\equiv \mathcal{Y}_{ijk}(\mathbf{X})\, dX^i \wedge dX^j \wedge dX^k = Y_{ijk}(\mathbf{x})\, dx^i \wedge dx^j \wedge dx^k\end{aligned}$$

Englert Equation reduces to a linear algebraic equation on coefficients

$$\begin{aligned}\sqrt{\det \mathcal{C}}\; \epsilon_{ijk}{}^{\ell mnp}\, k_\ell\, v_{mnp} &= -\frac{6\,\mu}{\pi}\, \omega_{ijk} \\ \sqrt{\det \mathcal{C}}\; \epsilon_{ijk}{}^{\ell mnp}\, k_\ell\, \omega_{mnp} &= \frac{6\,\mu}{\pi}\, v_{ijk}\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}\mu^2 &= \pi^2\, \|\,\mathbf{k}\,\|^2 \quad ; \quad \|\,\mathbf{k}\,\|^2 = k_\ell\, k_m \left(\mathcal{C}^{-1}\right)^{\ell m} \\ 0 &= \left(\mathcal{C}^{-1}\right)^{\ell m} k_\ell\, v_{ijm} = \left(\mathcal{C}^{-1}\right)^{\ell m} k_\ell\, \omega_{ijm}\end{aligned}$$

Counting parameters minus constraints we find that each solution contains:

# of parameters $= 20 \times$ (# of **k** in the orbit $\mathcal{O}$)

We need to classify weight lattice orbits

This is equivalent to the classification of conjugacy classes of subgroups of L_{168}

My findings about the available L_{168}-orbits in the weight lattice are summarized below

1. Orbits of length 8 (one parameter n; stability subgroup $\mathbf{H_s = G_{21}}$)
2. Orbits of length 14 (two types A & B) (one parameter n; stability subgroup $\mathbf{H_s = T_{12A,B}}$)
3. Orbits of length 28 (one parameter n ; stability subgroup $\mathbf{H_s = Dih_3}$)
4. Orbits of length 42 (one parameter n; stability subgroup $\mathbf{H_s = Z_4}$))
5. Orbits of length 56 (three parameters n,m,p; stability subgroup $\mathbf{H_s = Z_3}$)
6. Orbits of length 84 (three parameters n,m,p; stability subgroup $\mathbf{H_s = Z_2}$)
7. Generic orbits of length 168 (seven parameters stability subgroup $\mathbf{H_s = 1}$)

T_{12} = tetrahedral group

Dih_n = dihedral group of type n

$G_{21} \equiv \mathbb{Z}_3 \ltimes \mathbb{Z}_7$

This is the analogue of the classification of octahedral symmetric polyhedric figures in the cubic lattice.

The maximal subgroup G_{21}

Presentation $\quad \mathcal{X}^3 = \mathcal{Y}^7 = 1 \quad ; \quad \mathcal{X}\mathcal{Y} = \mathcal{Y}^2\mathcal{X}$

Conjugacy classes

ConjugacyClass	C_1	C_2	C_3	C_4	C_5
representative of the class	e	$\mathcal{Y}$	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
order of the elements in the class	1	7	7	3	3
number of elements in the class	1	3	3	7	7

character table

0	e	$\mathcal{Y}$	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	$\mathcal{X}$
$D_1\,[G_{21}]$	1	1	1	1	1
$DX_1\,[G_{21}]$	1	1	1	$-(-1)^{1/3}$	$(-1)^{2/3}$
$DY_1\,[G_{21}]$	1	1	1	$(-1)^{2/3}$	$-(-1)^{1/3}$
$DA_3\,[G_{21}]$	3	$\frac{1}{2}i\left(i+\sqrt{7}\right)$	$-\frac{1}{2}i\left(-i+\sqrt{7}\right)$	0	0
$DB_3\,[G_{21}]$	3	$-\frac{1}{2}i\left(-i+\sqrt{7}\right)$	$\frac{1}{2}i\left(i+\sqrt{7}\right)$	0	0

Construction of a G_{21} invariant Englert flux

$$\mathcal{O}_7 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

G_{21} orbit of 7 vectors in the weight lattice

$$\Delta = \{dX_1 \wedge dX_2 \wedge dX_3, dX_1 \wedge dX_2 \wedge dX_4, \ldots\ldots, dX_5 \wedge dX_6 \wedge dX_7\} = \{\Delta_{\mathfrak{q}}\}\ ,\ (\mathfrak{q} = 1, \ldots, 35)$$

The solution has the form

$$\mathbf{Y}(\mathbf{X}|\mathbf{F}) = \sum_{\alpha=1}^{14} \sum_{\mathfrak{q}=1}^{35} \mathfrak{C}_{\mathfrak{q}\alpha}(\mathbf{F})\, \Delta_{\mathfrak{q}} \times \mathfrak{f}_\alpha(\mathbf{X})$$

where $\mathfrak{C}_{\mathfrak{q}\alpha}(\mathbf{F})$ are linear combinations of the 140 parameters F_I

$$\mathfrak{f}_\alpha(\mathbf{X}) = \begin{pmatrix} \text{Cos}\,[2\pi X_1] \\ \text{Cos}\,[2\pi(-X_1+X_2)] \\ \text{Cos}\,[2\pi(-X_2+X_3)] \\ \text{Cos}\,[2\pi(-X_3+X_4)] \\ \text{Cos}\,[2\pi(-X_5+X_6)] \\ \text{Cos}\,[2\pi X_7] \\ \text{Cos}\,[2\pi(-X_6+X_7)] \\ \text{Sin}\,[2\pi X_1] \\ \text{Sin}\,[2\pi(-X_1+X_2)] \\ \text{Sin}\,[2\pi(-X_2+X_3)] \\ \text{Sin}\,[2\pi(-X_3+X_4)] \\ \text{Sin}\,[2\pi(-X_5+X_6)] \\ -\text{Sin}\,[2\pi X_7] \\ \text{Sin}\,[2\pi(-X_6+X_7)] \end{pmatrix}$$

Decomposition into irreps

$$\mathfrak{D}_{140}\,[G_{21}] \;=\; 8\,D_1\,[G_{21}] \oplus 20\,DA_3\,[G_{21}] \oplus 20\,DB_3\,[G_{21}] \oplus 6\,DX_1\,[G_{21}] \oplus 6\,DY_1\,[G_{21}]$$

Hence there is a G_{21} invariant Englert 3-form depending on 8 moduli $\Psi = \{ \psi_\alpha \}$

Transforming to the orthogonal basis and considering the operator

$$\mathcal{B}(\mathbf{x}|\mathbf{\Psi}) = \sum_{ijk} Y^s_{ijk}(\mathbf{x}|\mathbf{\Psi})\tau_{ijk}$$

We find that it has generically rank 8. There are however 4 constraints that reduce its rank to 7 and they are unique.
Further reduction of the rank is not possible.

Hence we have found a 4-parameter N=1 M2-brane solution with Englert fluxes that is exactly invariant against the non abelian group G_{21}

A taste of the solution.....

$$\mathbf{Y}^{\mathcal{N}=1}(\mathbf{x}|\psi) = \sum_{\alpha=1}^{14} \sum_{\mathfrak{q}=1}^{35} \mathfrak{C}^{\mathcal{N}=1}_{q\alpha}(\psi)\, \mathbf{d}_{\mathfrak{q}} \times \widehat{f}_{\alpha}(\mathbf{x})$$

$$\mathbf{d} = \{dx_1 \wedge dx_2 \wedge dx_3, dx_1 \wedge dx_2 \wedge dx_4, \ldots\ldots, dx_5 \wedge dx_6 \wedge dx_7\} = \{\mathbf{d}_{\mathfrak{q}}\}$$

$$\hat{f}_\alpha(\boldsymbol{x}) = \begin{pmatrix}
\mathrm{Cos}\left[2\pi\left(\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(-\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(-\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{3x_6}{2\sqrt{2}} - \sqrt{2}x_6 - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(-\frac{x_3}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(\frac{x_4}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Cos}\left[2\pi\left(\frac{x_4}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(-\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(-\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{3x_6}{2\sqrt{2}} - \sqrt{2}x_6 - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(-\frac{x_3}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
-\mathrm{Sin}\left[2\pi\left(\frac{x_4}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}}\right)\right] \\
\mathrm{Sin}\left[2\pi\left(\frac{x_4}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}}\right)\right]
\end{pmatrix}$$

The coefficients $\mathfrak{C}^{\mathcal{N}=1}_{q\alpha}(\boldsymbol{\psi})$ are linear combinations of the 4 parameters $\psi_{1,2,3,4}$ but displaying them is too much ponderous

They considerably simplify in the points

$$\vec{\psi}_1 \equiv \{\psi_1 = 1, \psi_2 = -1, \psi_3 = -2, \psi_4 = 1\}$$
$$\vec{\psi}_2 \equiv \left\{\psi_1 = 1, \psi_2 = -1, \psi_3 = \frac{1}{2}, \psi_4 = 1\right\}$$

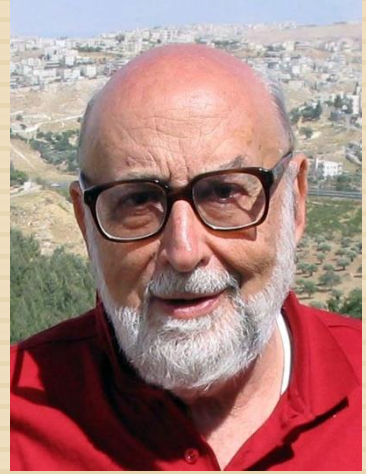
There we can explicitly calculate the warp factor.

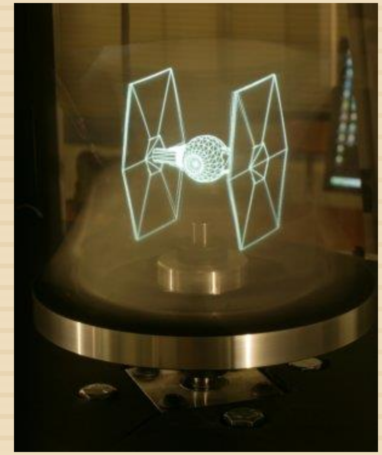
Non linear structure in d=3

$$
\begin{aligned}
&\mathfrak{W}(x) = \\
&21\,\mathrm{Cos}\left[2\sqrt{2}\pi x_1\right] + 21\,\mathrm{Cos}\left[2\sqrt{2}\pi x_2\right] + 21\,\mathrm{Cos}\left[2\sqrt{2}\pi x_4\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 - x_4 - x_5 - x_6)\right] \\
&+21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 + x_4 - x_5 - x_6)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 + x_3 + x_5 - x_6)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 - x_3 - x_5 + x_6)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 - x_4 + x_5 + x_6)\right] \\
&+21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 + x_4 + x_5 + x_6)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 - x_2 + x_5 - x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 + x_2 + x_5 - x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 - x_4 - x_6 - x_7)\right] \\
&+21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 + x_4 - x_6 - x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 - x_3 + x_6 - x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 - x_2 - x_5 + x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 + x_2 - x_5 + x_7)\right] \\
&+21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_3 - x_4 + x_5 + x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_3 + x_4 + x_5 + x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_1 + x_3 - x_6 + x_7)\right] + 21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 - x_4 + x_6 + x_7)\right] \\
&+21\,\mathrm{Cos}\left[\sqrt{2}\pi\,(x_2 + x_4 + x_6 + x_7)\right] + \sqrt{7}\mathrm{Sin}\left[2\sqrt{2}\pi x_1\right] - \sqrt{7}\mathrm{Sin}\left[2\sqrt{2}\pi x_2\right] - \sqrt{7}\mathrm{Sin}\left[2\sqrt{2}\pi x_4\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 - x_4 - x_5 - x_6)\right] \\
&+\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 + x_4 - x_5 - x_6)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 + x_3 + x_5 - x_6)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 - x_3 - x_5 + x_6)\right] \\
&+\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 - x_4 + x_5 + x_6)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 + x_4 + x_5 + x_6)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 - x_2 + x_5 - x_7)\right] \\
&+\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(-x_1 + x_2 + x_5 - x_7)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 + x_2 + x_5 - x_7)\right] + \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 - x_4 - x_6 - x_7)\right] \\
&+\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 + x_4 - x_6 - x_7)\right] + \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 - x_3 + x_6 - x_7)\right] \\
&+\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 + x_2 - x_5 + x_7)\right] + \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_3 - x_4 + x_5 + x_7)\right] + \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_3 + x_4 + x_5 + x_7)\right] \\
&-\sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_1 + x_3 - x_6 + x_7)\right] - \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 - x_4 + x_6 + x_7)\right] + \sqrt{7}\mathrm{Sin}\left[\sqrt{2}\pi\,(x_2 + x_4 + x_6 + x_7)\right]
\end{aligned}
\tag{9.27}
$$

$$
H(U,x) = \alpha - 2268\, e^{4\sqrt{14}\pi U} + \frac{16}{3} e^{4\sqrt{14}\pi U}\, \mathfrak{W}(x)
$$

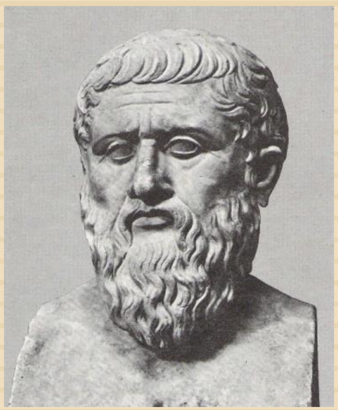
The supersymmetric d=3 theory has clearly complicated non linear interactions governed by its discrete symmetry……….





M2-branes

$G_{21} \subset PSL(2, \mathbb{Z}_7)$



N=1 Theory in D=3 with discrete symmetry G_{21}

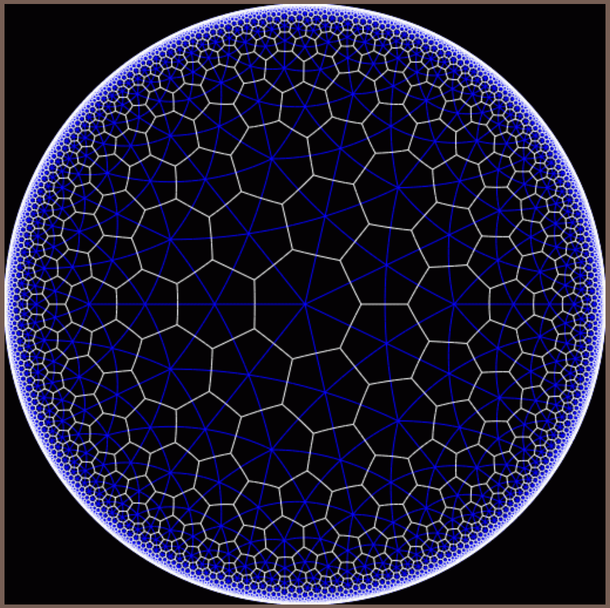
CONCLUSIONS = OPEN QUESTIONS

We have discovered supersymmetric M2 –branes with Englert fluxes and singled out a challenging new crystallographic system in d=7

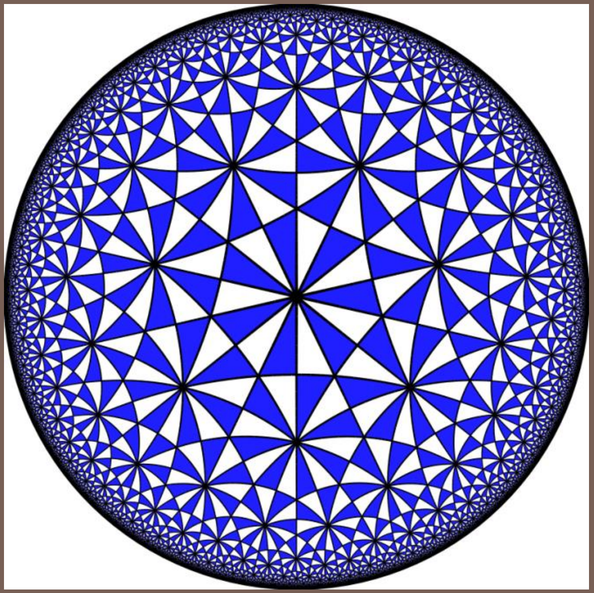
The game has just only started. We have now to analyse the very rich consequences of these constructions. There are many open questions

Just some of the open questions

- Which invariants discriminate among supersymmetric and non supersymmetric moduli?
- What might we obtain from higher orbits?
- Can we associate Polyhedra to the PSL(2,7) orbits in the weight lattice?
- What is the relation with the Klein Quartic?
- Can we relate our crystallographic constructions to G_2-manifolds?
- How to use the discrete symmetry in the calculation of d=3 quantum correlators?
- ………..



7 ⟷ 3



THANK YOU VERY MUCH FOR YOUR ATTENTION