Episteme e Technè: da Archimede al XXI secolo

I flussi di Arnold-Beltrami come un esempio paradigmatico

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Sextus Empyricus

In his essay "Against the Rhetors" (Πρὸς ῥητορικούς) the sceptic philosopher Sextus Empiricus (II century A.D) distinguishes Science (ἐπιστήμη) from Technique (τέχνη). He says:

> Every τέχνη is a system of knowledges put together in order to pursue a goal useful for life.

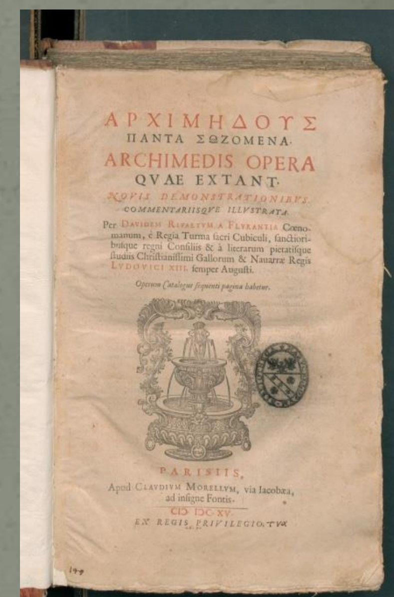
The main idea is the ***Utilitarism*** intrinsic to ***Technology***, whose absence is instead intrinsic to the very definition of ***Science*** (ἐπιστήμη). Yet without ἐπιστήμη there is no τέχνη

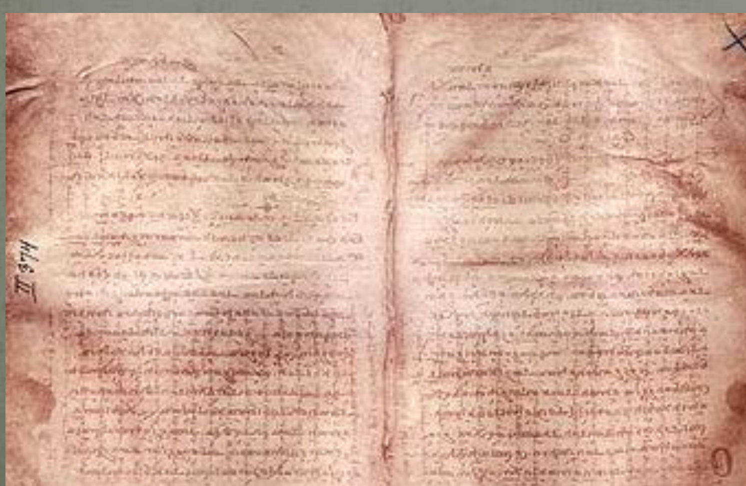
The concept of ***Technological Transfer*** was clear to Sextus Empiricus who lived about 1900 years ago!

Archimedes 287 BC – 212 BC Syracuse

Archimedes wrote his scientific essays and letters in the doric dialect spoken in Syracuse. Isidore of Miletus was a renowned scientist and mathematician before Emperor Justinian hired him as an Architect to construct Hagia Sophia. It was in the school of Isidorus that the two most famous Treatises of Archimedes surviving in the Middle Ages were turned from doric into κοινὴ διάλεκτος.

In the ninth century Leon, who restored the University of Constantinople, collected together all the works of Archimedes: ***Codex A, Gothic B and Codex C.***





Codex C was transformed into a palimpsest by Johannes Myronas in 1229. The palimpsest was discovered by the Danish scholar Heiberg in 1906.

About Archimedes

In the life of Marcellus, Plutarch wrote:

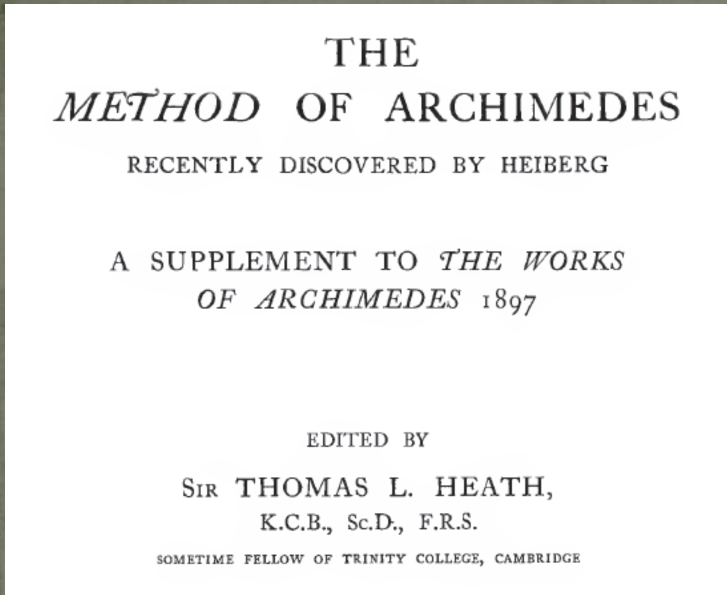
"He placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life."

Yet Archimedes is the inventor of what we can name *experimental mathematics*

In the essay ***The Method (On the Mechanical Theorems*** (Περί τῶν μηχανικῶν θεωρήματων)) which was a letter to Eratosthenes :

*………Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer of mathematical inquiry, I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the **problems in mathematics by means of mechanics**. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves ; for certain things first became clear to me by a mechanical method, **although they had to be demonstrated by geometry afterwards** because their investigation by the said method did not furnish an actual demonstration.*

Translation by Sir Thomas Heath in 1912, six years after the discovery of the Palimpsest



THE

METHOD OF ARCHIMEDES

RECENTLY DISCOVERED BY HEIBERG

A SUPPLEMENT TO *THE WORKS OF ARCHIMEDES* 1897

EDITED BY

SIR THOMAS L. HEATH,

K.C.B., Sc.D., F.R.S.

SOMETIME FELLOW OF TRINITY COLLEGE, CAMBRIDGE

The role of the Byzantine scholar Basilios Bessarion (1403-1473) who became a Roman Catholic Cardinal in preserving the Byzantine Heritage of Greek Culture and of its sources was considerable. Bessarion owned a copy of Archimedes works.

Basilios Bessarion

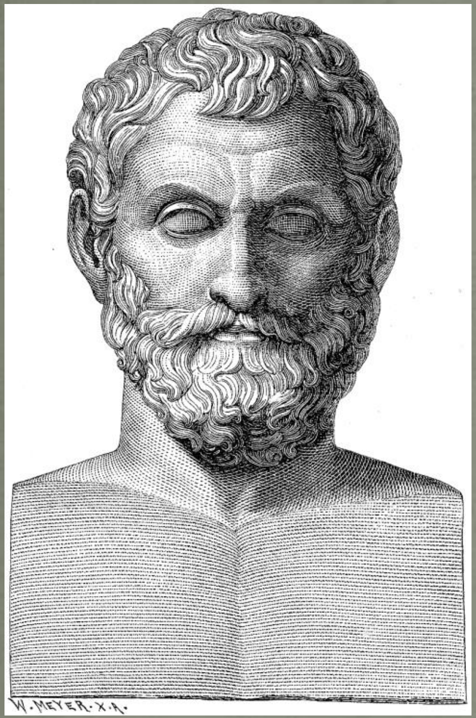
His books donated to Venice are the initialcore of the Marcian Library.

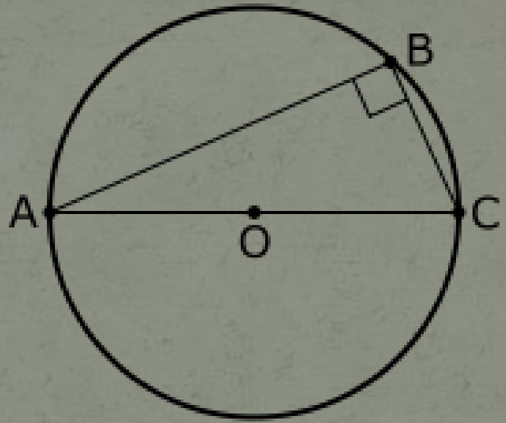
Archimedes treaty on the Quadrature of the Parabola contains the first example of the sum of an infinite series

$$\sum_{n=0}^{\infty} 4^{-n} = \frac{4}{3}$$

Archimedes **method of exhaustion** is the ancient craddle of infinitesimal calculus to be developed by Newton and Leibnitz 19 century later

Geometry was born in the East





Thales theorem .

According to tradition Thales Milesius, the first Greek philosopher and mathematician, learnt about geometrical formulae in Egypt and in Babylon but he was the first who conceived the deductive method and provided what we call a mathematical proof of a theorem. Indeed the proposition about the inscription of a rectangular triangle in a circle.

Mathematics and Democracy

In his impressive Essay on Hellenic and Hellenistic Science, Lucio Russo[153] puts into evidence the role in the development of the conception of a mathematical proof that was played by the practice of democracy in the Greek $\pi \acute{o} \lambda \iota \varsigma$ and by the frequent need of the Greek citizen, the $\pi o \lambda \acute{\iota} \tau \eta \varsigma$, to advocate publicly his own case in front of juries.

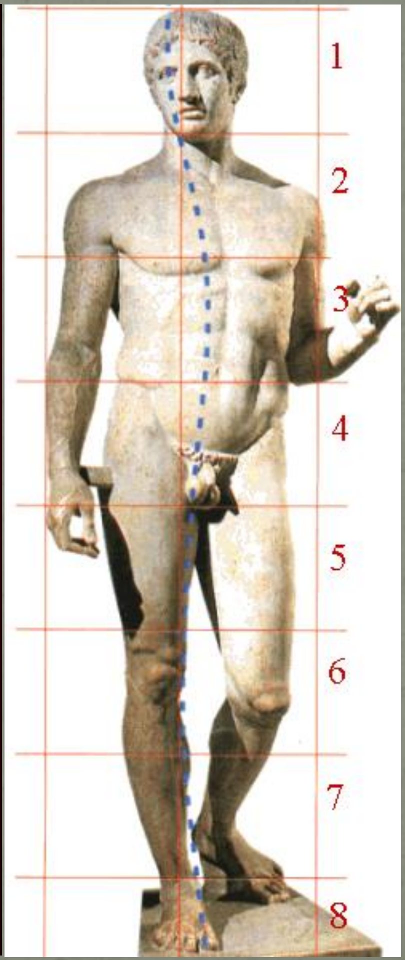
The Greek word for proof, $\alpha \pi \acute{o} \delta \varepsilon \iota \xi \iota \varsigma$, comes from the verb $\alpha \pi o \delta \varepsilon \acute{\iota} \kappa \nu \upsilon \mu \iota$ which means *I present, I submit*. Russo illustrates the close relation between the geometrical $\alpha \pi \acute{o} \delta \varepsilon \iota \xi \iota \varsigma$ and Rhetoric, quoting Aristotle's *Ars Rhetorica* where the *enthymemes* of rhetors are shown to be just *syllogisms*: then he recalls the impressive sentence by Quintilian: …*nullo modo sine geometria esse possit orator*, there can be no orator without geometry[148].

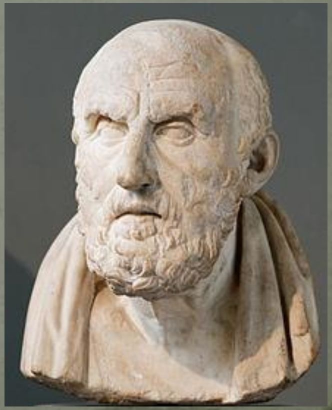
Quotations from my book published by Springer:

A Conceptual History of Space and Symmetry: from Plato to the SuperWorld

Polykleitos Canon

The Canon, was the title of a treatise written by the Vth century sculptor Polykleitos, who exemplified his theory in a bronze statue, *the Doryphoros* (the Spear Bearer). Both the treatise and the statue lost, but a roman marble copy dating about 120 BC has reached us from Pompeii and it is preserved in Naples National Archaeological Museum. A quotation from the treatise has survived in the book *De Placitis Hippocratis et Platonis* by Galen, the famous medical writer of the II century A.D..

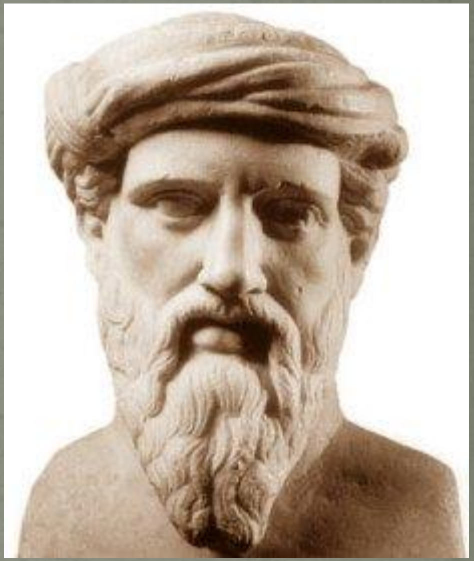




Symmetry meant simple rational ratios among the dimensions of the parts

in his METAPHYSICS Aristotle says:

. . . the so-called Pythagoreans applied themselves to the study of mathematics, and were the first to advance that science; insomuch that, having been brought up in it, they thought that its principles must be the principles of all existing things.



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Aristotle observes that the ONE is reasonably regarded as not being itself a number, because a measure is not the things measured, but the measure or the ONE is the beginning (or principle) of number.

Symmetric (or harmonic) meant:

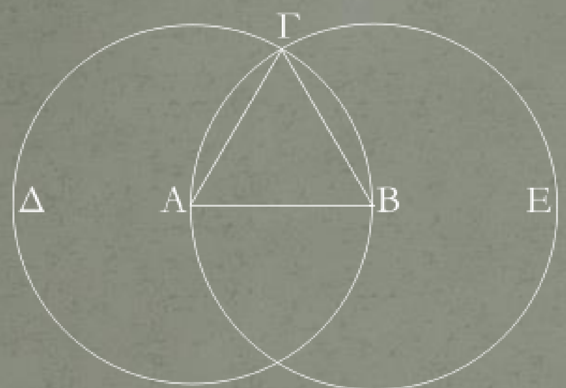
Easily countable,
hence understandable,
therefore beautiful!

$\sqrt{2}$

In geometry however made their appearance irrational numbers and Pythagorean philosophy entered a crisis

Euclid’s Book is a Review of three Centuries of Math. Research

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.
Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.
Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.



Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ᾽ ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. ὅπερ ἔδει ποιῆσαι.

Euclid of Alexandria was active during the reign of Ptolemy I (323 - 283 BC).
In the picture we see Euclid as imagined by Raffaello in his School of Athens and on the side an example of a proof from the original Greek version of the Elements.

History of a Book

The Elements is probably the most famous mathematical textbook of all times.

The Greek original was transmitted through the edition cured by Theon from Alexandria in the IV century AD. In 1808 Francois Peyrard discovered in Vatican a manuscript of the Elements coming from a byzantine workshop of the Xth century that was not based on Theon's edition. The first latin translation appears to have been produced by Boethius in the VI century AD but then the Elements disappeared in Western Europe, until the English monk Adelard of Bath produced a Latin translation of an Arabic version.

The Arabs received the Elements from the Byzantines approximately around 760; this version was translated into Arabic under Harun al Rashid c. 800 and became the source of Adelard. Theon's Greek edition was recovered in 1533.

From XIIIth century Alhambrain Grenada

from a finite portion of the figure we can predict it in the entire plane

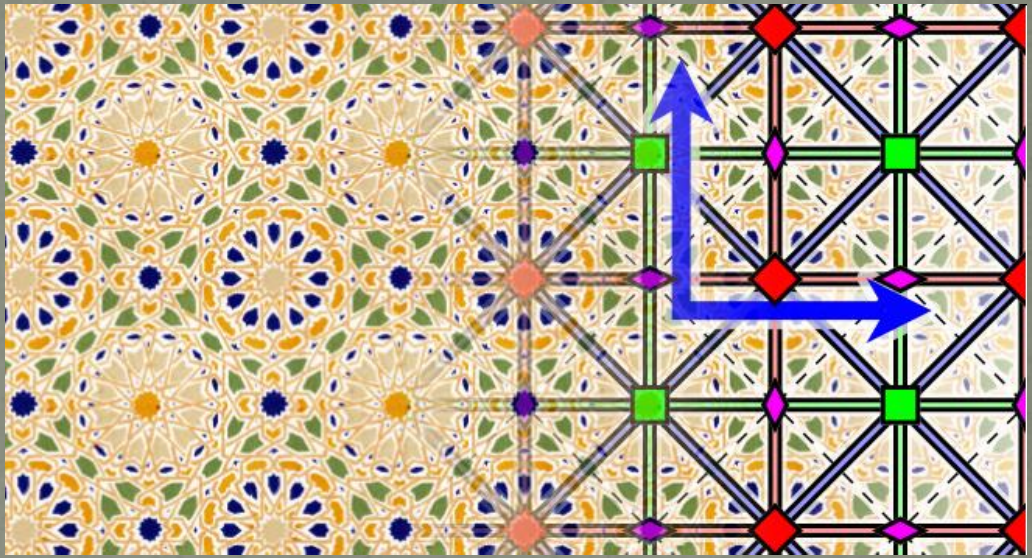
How many of these patterns are possible?

ANSWER 17!

All of these 17 patterns are represented in Alhambra, yet the reason behind was understood only in 1891 from Fedorov in Sankt Peterbourg :

TESSELLATION GROUPS of the PLANE!

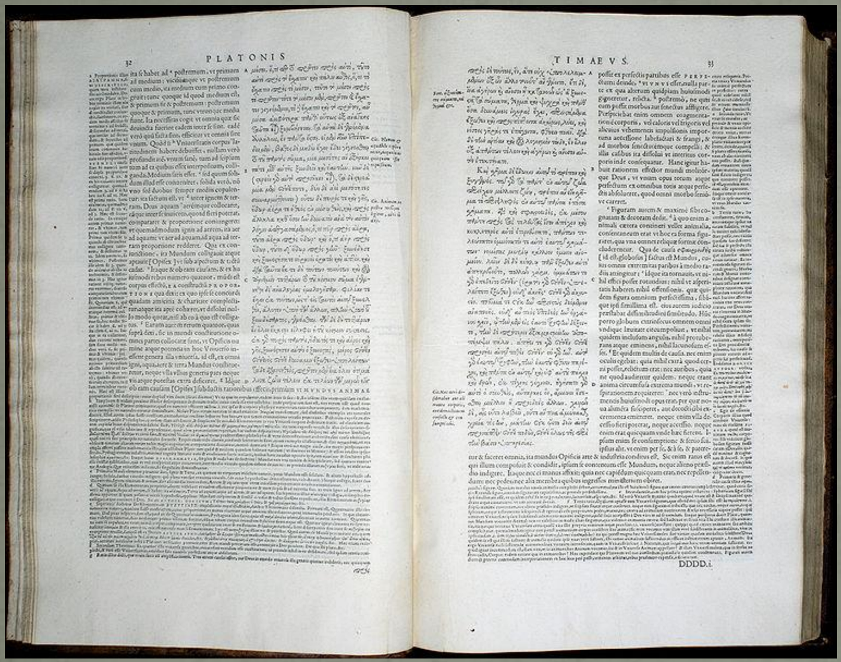
Mathematics and Crystallography



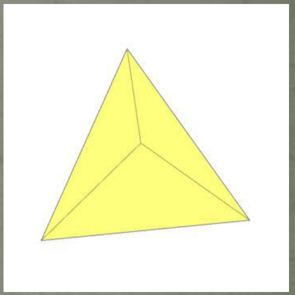
Evgraf Stepanovich Fyodorov (Orenburg 1853 – Sankt Peterburg 1919).

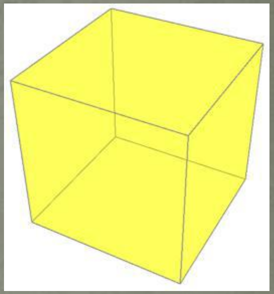
The key word is **GROUP!** We need to go back to the beginning of the XIX century: **GALOIS and still before, PLATO**

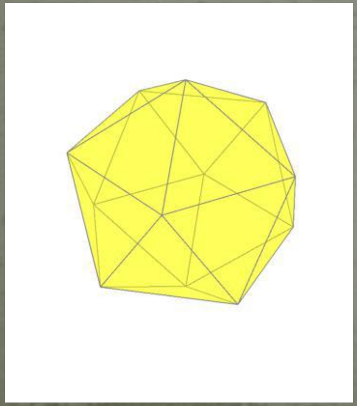
Plato's Timaeus (circa 360 B.C.)

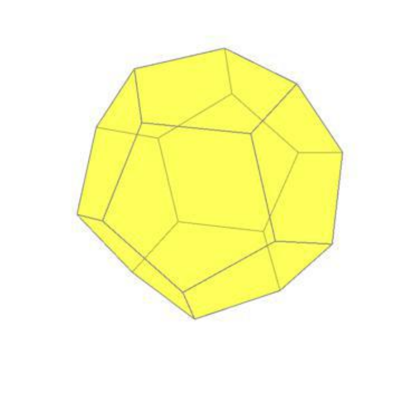


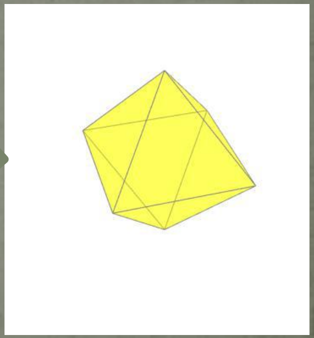
Timaeus includes the first mention of five regular solids and the platonic theory of fundamental constituents of matter.



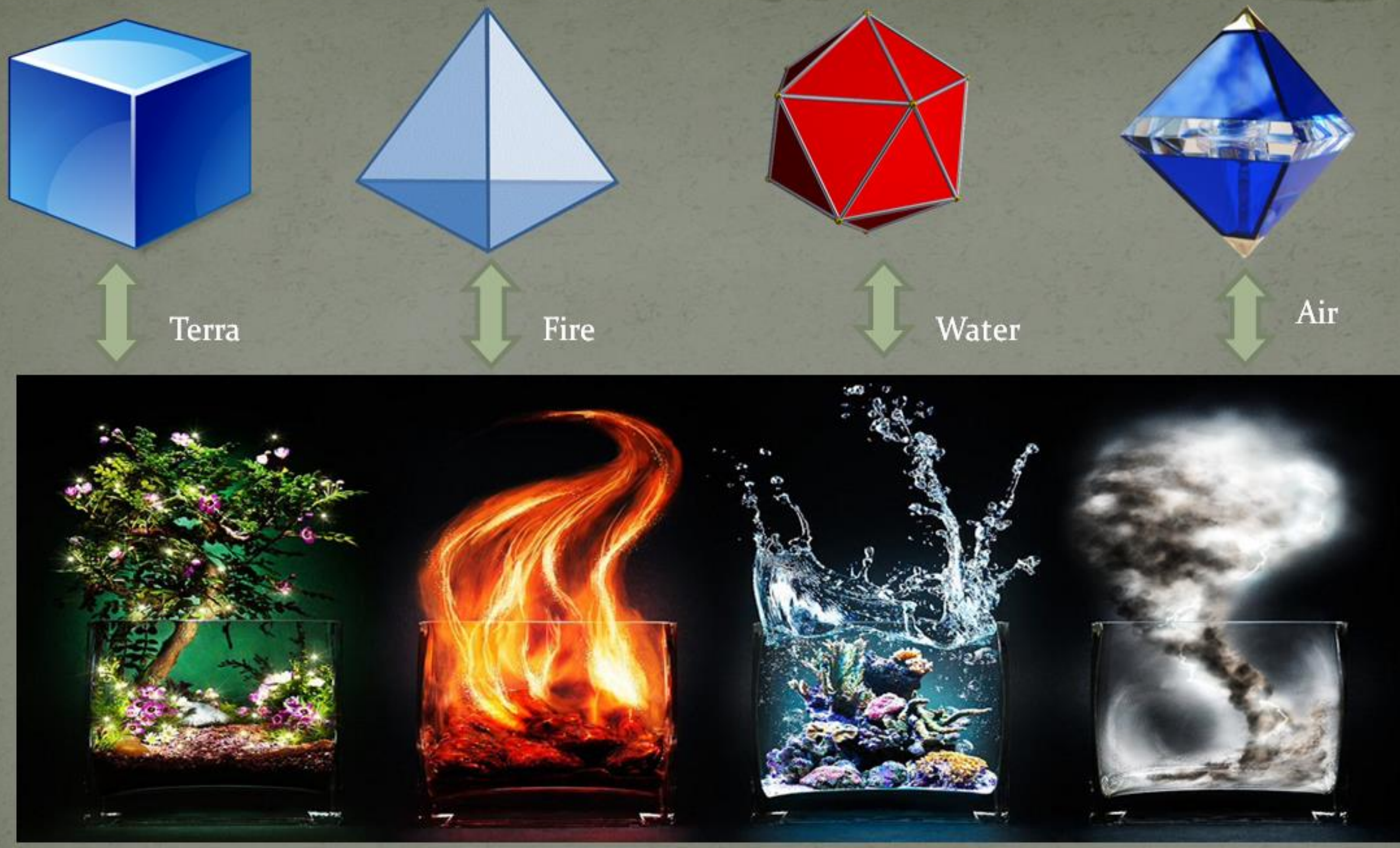




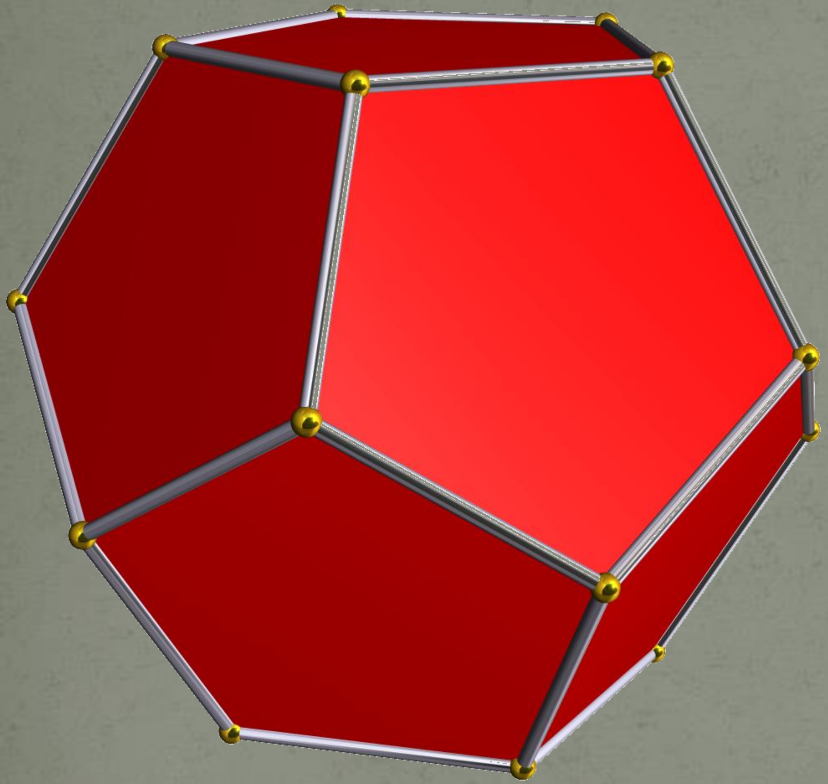




The 4 elements

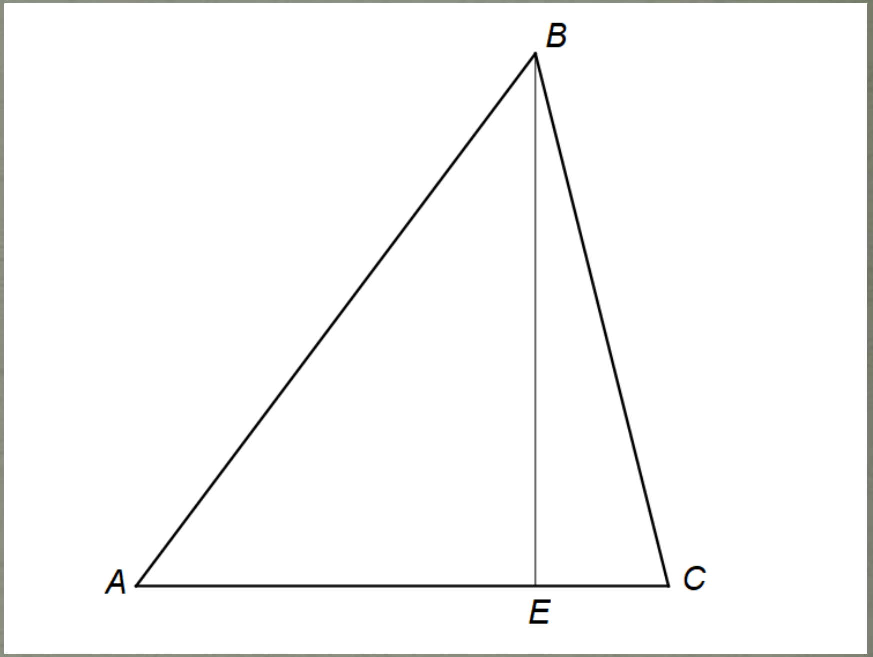


The spirit, aether or quintessence



Ante litteram Regge calculus of Timaeus

Thus Timaeus argues that all the elements are three dimensional bodies that occupy a finite portion of space. Such finite volumes correspond to some solids delimited by faces that are, on their turn, finite portions of the plane.



All plane figures and in particular the faces of the considered solid can be triangulated, namely they can be decomposed into triangles. All triangles, on their turn, can be decomposed into two rectangular triangles as shown.

Ante litteram Regge calculus of Timaeus

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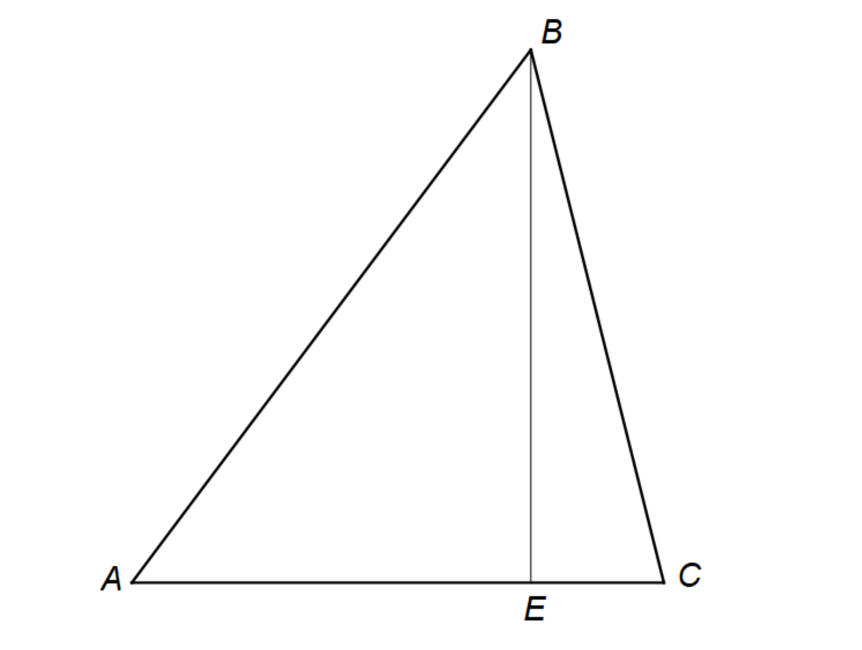

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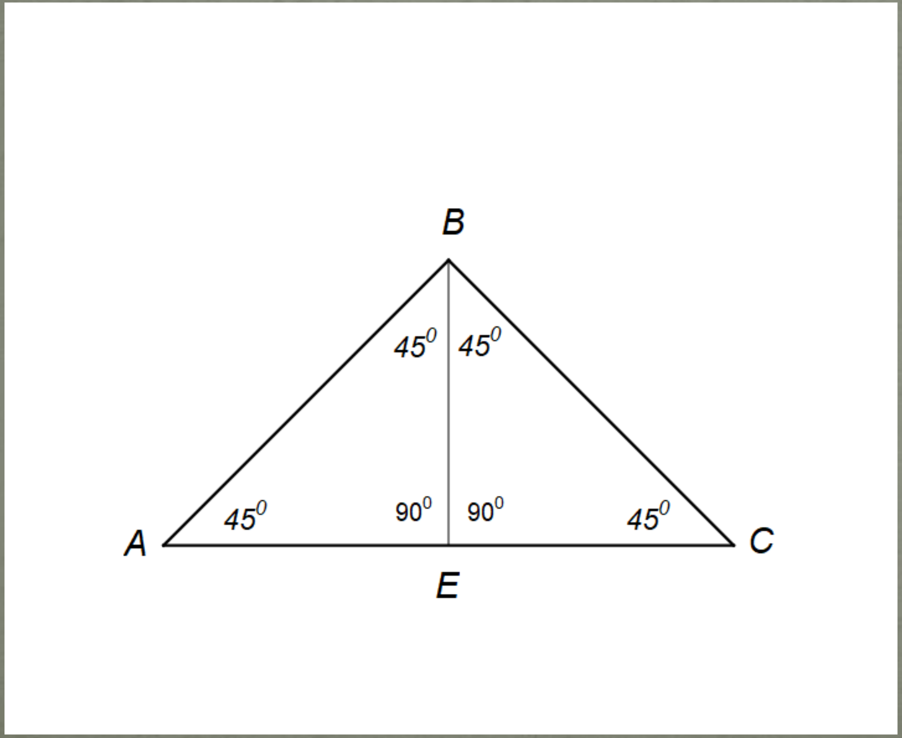
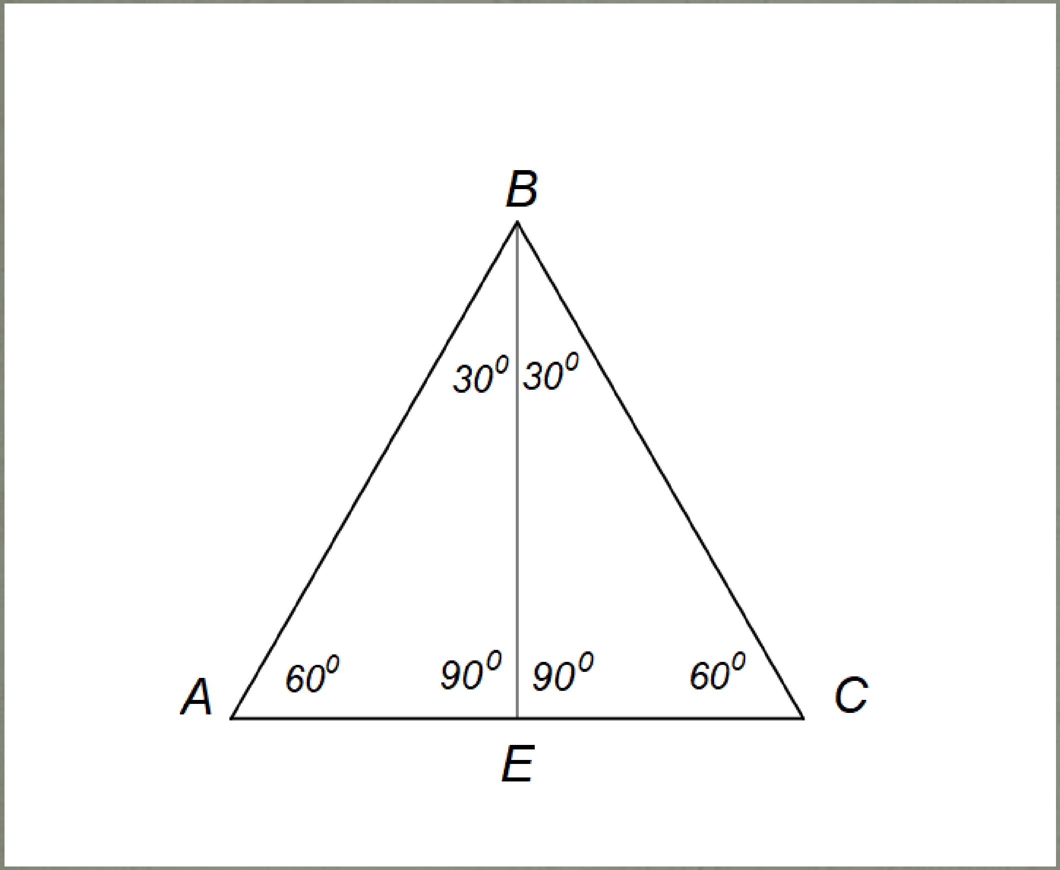
Timaeus says:

These constituent triangles, then, proceeding by a combination of probability with demonstration, we assume to be the original elements of fire and the other bodies ; but the principles which are prior to these God only knows, and he of men who is the friend of d God



Unique type of isosceles rectagular triangle

The most beautiful scalene triangle



Now, the one which we maintain to be the most beautiful of all the many triangles (and we need not speak of the others) is that of which the double forms a third triangle which is equilateral

Plato's views on Mathematics

The question is whether Plato, admired by the modern founders of physics Copernicus, Kepler and Galilei, as the ancient philosopher who shared their belief in mathematics as the language of Nature, was actually committed to a mathematical theory of the world.

There are several hints that he was not. In the Republic he says

Motion presents not just one, but many forms.

One is that is imperfectly illustrated by celestial motions. The other is the musical motion, studied by Pythagorean acoustics.

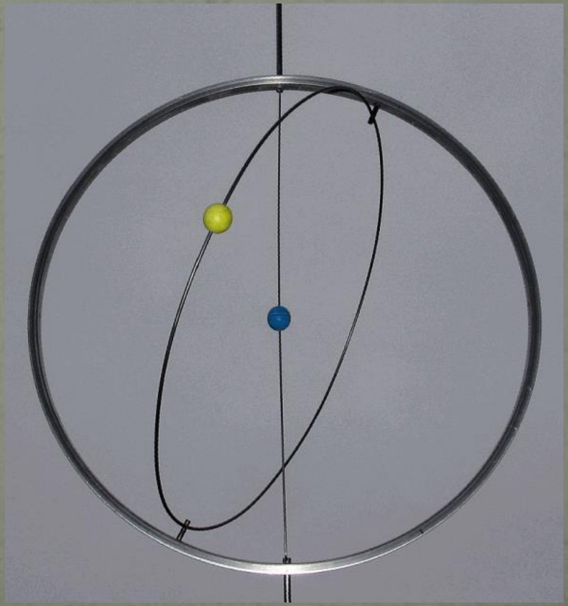
Eudoxus

Εὔδοξος ὁ Κνίδιος

We have Plato's warning to would-be astronomers, that they should not expect heavenly bodies to be excessively punctual, nor spend too much effort observing them in order to grasp their truth. It was probably aimed at none other than the young Eudoxus, who, while the Republic was being written, attended Plato's lectures and perhaps mentioned his plan for a mathematical theory of planetary motions

The life span of Eudoxus of Cnidus is probably (408-355 B.C.) and it is generally believed that his work is the ground basis of Euclid's Vth Book,

He introduced models of the heavenly motions based on uniformly rotating spheres whose poles, at the extremity of the rotation axes, are pinned on other rotating spheres. Such models evolved in the Ptolemaic theory of the cycles and epicycles



The Demiurge

καλοκᾰγαθὸς

The Demiurge, Creator of the Universe, was by definition the Very Good One. The Very Good One nothing is allowed to do that is not the most beautiful. Hence the Demiurge made the World beautiful, which, in line with the Canon of Polykleitos means in good proportions.

Demiurge's proportions

Translating into modern physical terms, the Demiurge created the Universe out of radiation (=fire) and matter (=earth) since what is generated has to be felt (=gravity) and it has to be seen (you need light). Yet in order to fulfil the imperative of beauty you need proportions and this implies at least one mean x as to be able to write **$F/X = X/E$** where F stands for fire and E for earth (this is certainly reminiscent of Eudoxus' theory of magnitude comparisons exposed in Euclid's Vth book).

However, the Universe had to be three-dimensional, rather than two-dimensional.

In $d = 3$ one mean is not sufficient to write proportions of solid bodies so you nead two means X;Y so to be able to write write

$F/X = X/Y = Y/E$.

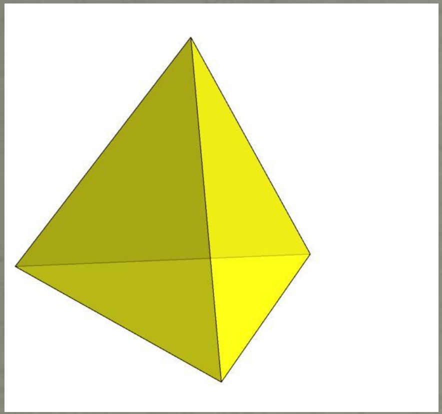
The **X and the Y** were the two additional elements **Air** and **Water**.

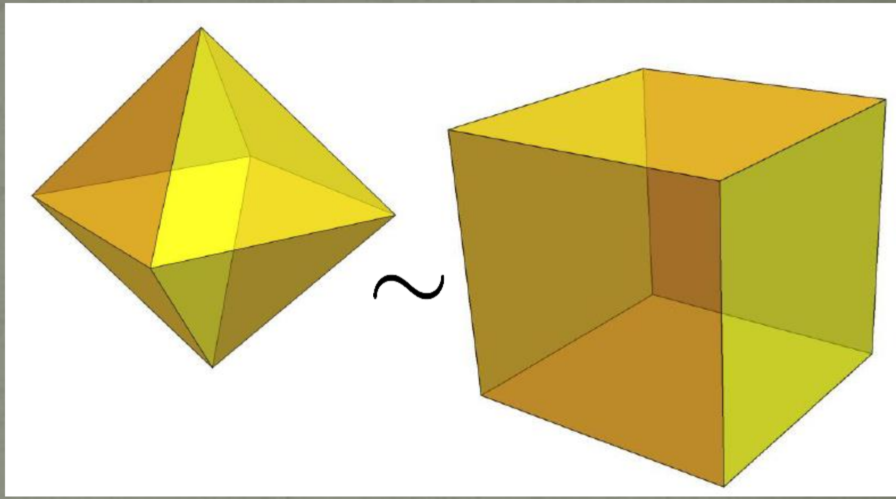
Antrum Platonicum

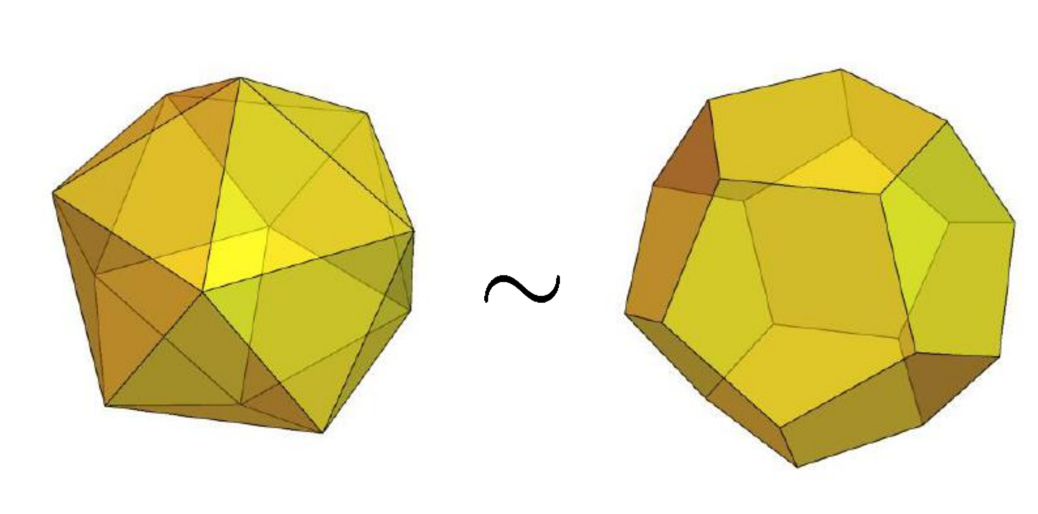
The argument that leads to the identification of the four elements with four out of the five regular solids is quite elaborate and based on the fundaments of Platonic Idealism.

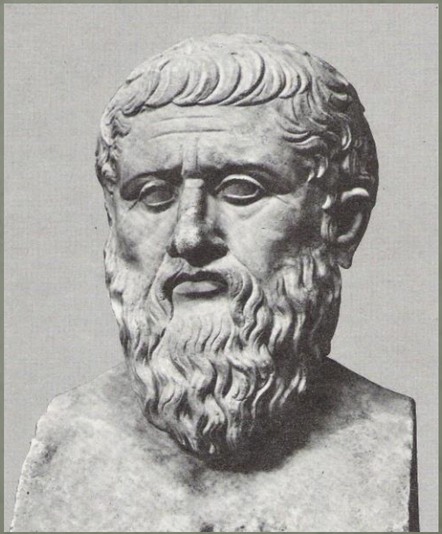
The patterns are the **Ideas**, the imitations of the patterns are the **Physical Phenomena**, the receptacle is just **Space**, the space of Geometry.

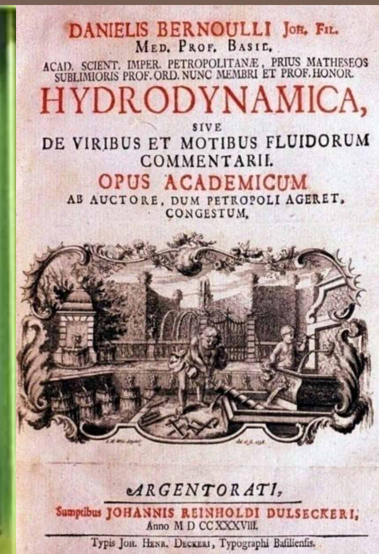
Platonic Groups





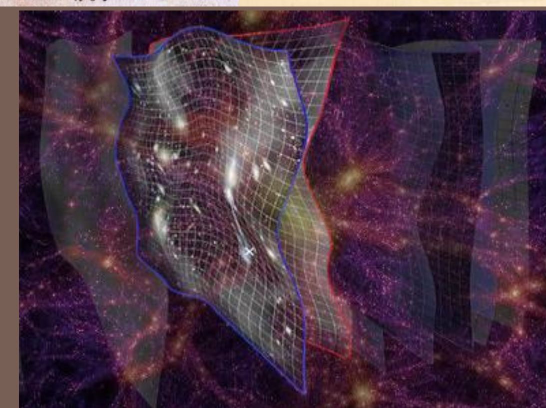






DANIELIS BERNOULLI Joh. Fil.
Med. Prof. Basil.
ACAD. SCIENT. IMPER. PETROPOLITANÆ, PRIUS MATHESEOS SUBLIMIORIS PROF. ORD. NUNC MEMBRI ET PROF. HONOR.
HYDRODYNAMICA,
SIVE
DE VIRIBUS ET MOTIBUS FLUIDORUM COMMENTARII.
OPUS ACADEMICUM
AB AUCTORE, DUM PETROPOLI AGERET, CONGESTUM.

ARGENTORATI,
Sumptibus JOHANNIS REINHOLDI DULSECKERI,
Anno M D CC XXXVIII.
Typis Joh. Henr. Deckeri, Typographi Basiliensis.



ARNOLD-BELTRAMI FLOWS & FLUXES

Euler equation of classical hydrodynamics

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p \quad ; \quad \nabla\cdot\mathbf{u} = 0$$

Mathematical Hydrodynamics

$$\mathcal{S} : \mathbb{R}_t \to \mathcal{M}_g$$

A flow is a smooth map from the time line to a Riemannian manifold

$$\forall t \in \mathbb{R} : u^i(x,t)\,\partial_i \equiv \mathsf{U}(t) \in \Gamma\,(T\mathcal{M}\,,\,\mathcal{M})$$

The velocity field is a section of the tangent bundle

$$\forall t \in \mathbb{R} : \Omega^{[\mathsf{U}]}(t) \equiv g_{ij}\,u^i(x,t)\,dx^j \in \Gamma\,(T\mathcal{M}\,,\,\mathcal{M})$$

lowering the indices we have a 1-form

Rewriting of Euler equations we obtain

$$-\,\mathrm{d}H = \partial_t\Omega^{[\mathsf{U}]} + \mathrm{i}_{\mathsf{U}}\cdot\mathrm{d}\Omega^{[\mathsf{U}]}$$

$$H = \left(p + \tfrac{1}{2}\parallel \mathsf{U} \parallel^2\right)$$

If H depends on x $\in$ M, the streamlines occur on level surfaces H=const and are two-dimensional. A necessary condition for chaos is

$$0 = \mathrm{i}_{\mathsf{U}}\cdot\mathrm{d}\Omega^{[\mathsf{U}]}$$

On a 3-torus

$$T^3 = \frac{\mathbb{R}^3}{\Lambda}$$

$$\star\, \mathrm{d}\mathbf{Y} = 2\,\pi\,\mu\,\mathbf{Y}$$

Beltrami Equation

Arnold Theorem

There are only two possibilities

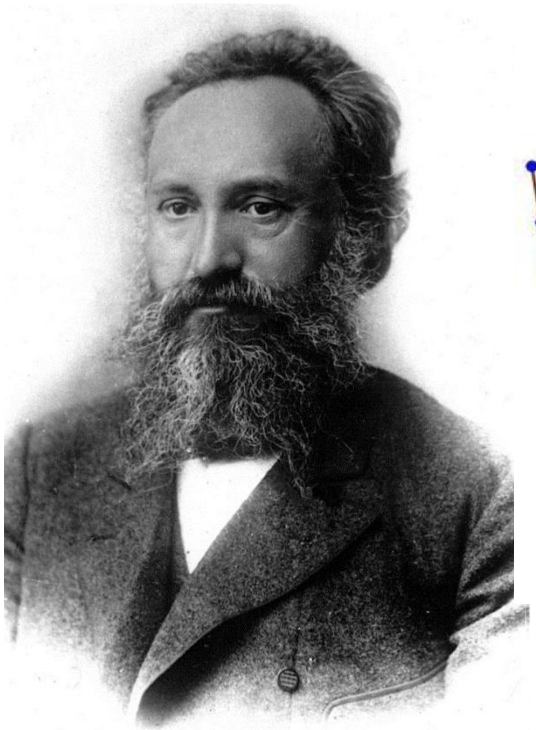
a) Either the form $\Omega^{[U]}$ is an eigenstate of the Beltrami operator $\star_g \mathrm{d}$ with a non vanishing eigenvalue $\lambda \neq 0$

$$\star_g\, \mathrm{d}\Omega^{[U]} = \lambda\, \Omega^{[U]}$$

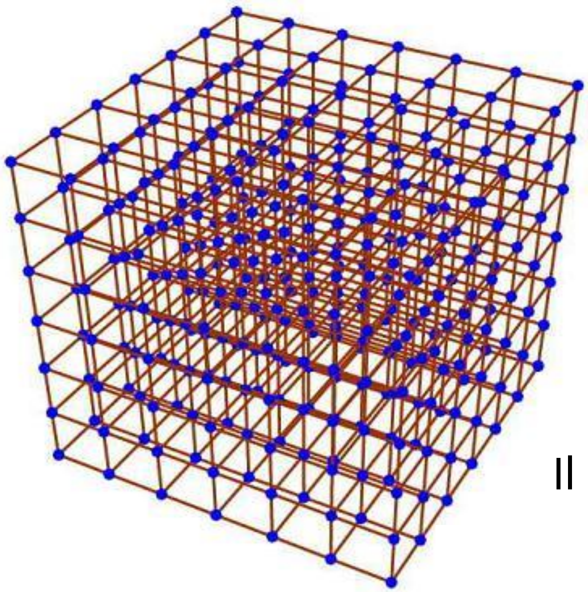
b) or the manifold $\mathcal{M}$ is subdivided into a finite collection of cells, each of which admits a foliation diffeomorphic to $T^2 \times \mathbb{R}$ and every two-torus T^2 is an invariant set with respect to the action of the velocity field U: in other words, all trajectories lay on some T^2 immersed in the manifold $\mathcal{M}$.

Two-dimensional streamlines = no chaos

Evgraf Federov e i gruppi cristallografici

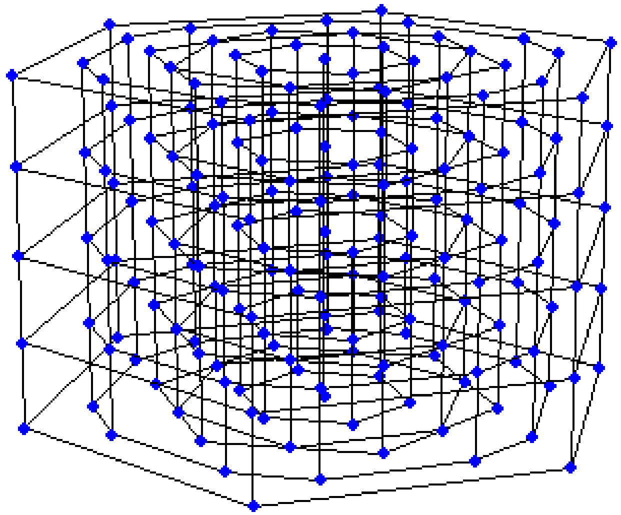


Evgraf Stepanovich Fedorov
1853 - 1920

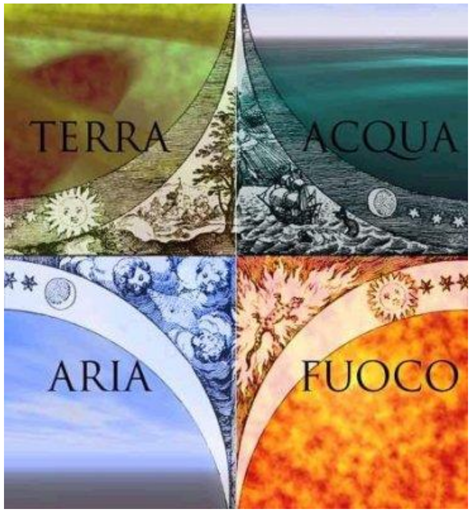


Il reticolo cristallino cubico

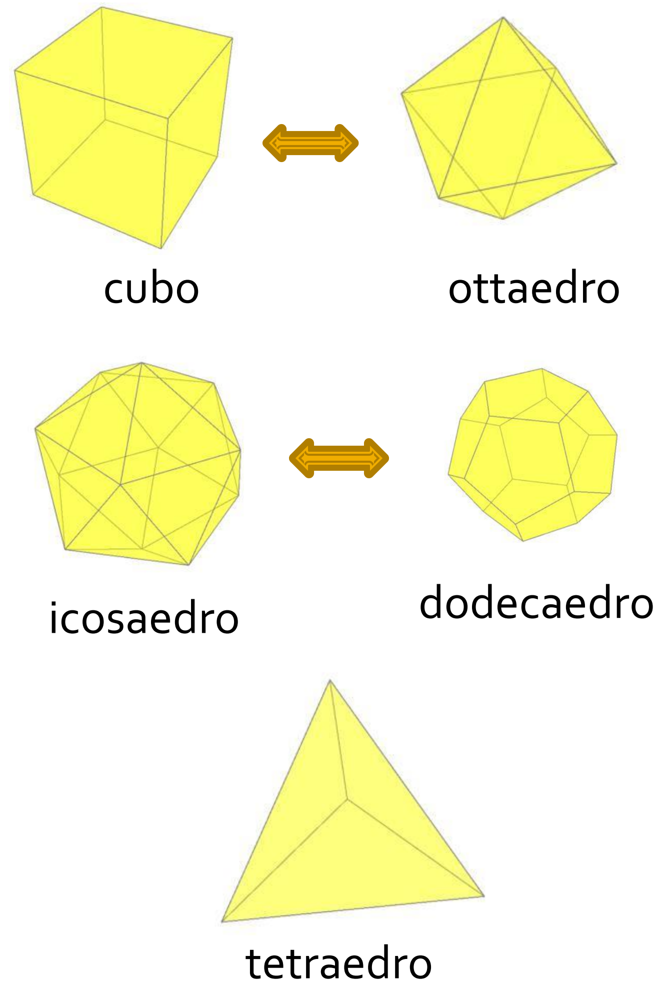
Il reticolo cristallino esagonale

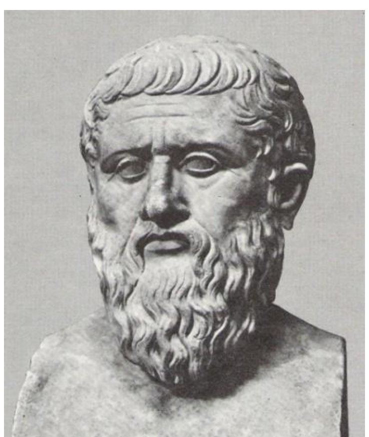


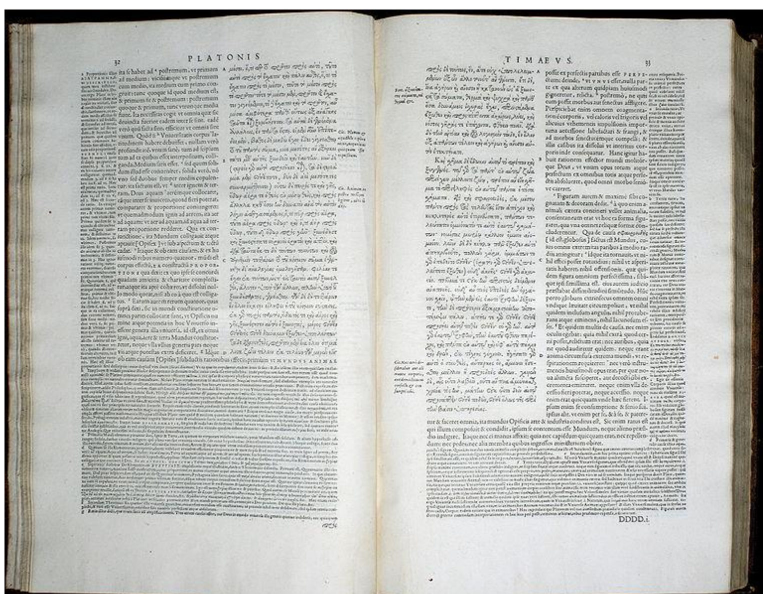
Ricordiamo il Timeo di 2350 anni fa….e poi….



La prima menzione dei cinque solidi regolari appare nel Timeo di Platone circa 350 A.C.
Il filosofo cerco' di spiegare l'universo associando i 4 elementi a 4 dei cinque solidi.

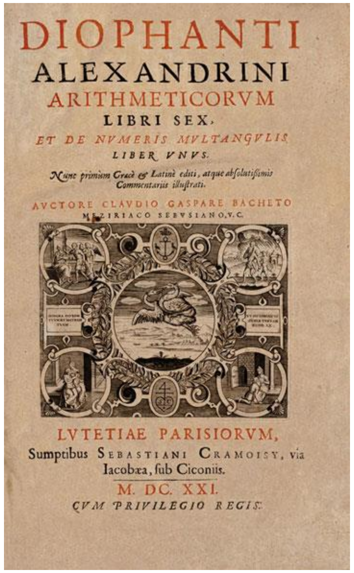






Andiamo ora a 1700 anni fa e poi al… XX secolo!

Diofanto di Alessandria
il padre dell'Algebra,
visse probabilmente nel
III secolo dopo Cristo.



DIOPHANTI ALEXANDRINI ARITHMETICORVM LIBRI SEX, ET DE NVMERIS MVLTANGVLIS LIBER VNVS.

Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.

AVCTORE CLAVDIO GASPARE BACHETO MEZIRIACO SEBVSIANO, V.C.

LVTETIAE PARISIORVM, Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.

M. DC. XXI.

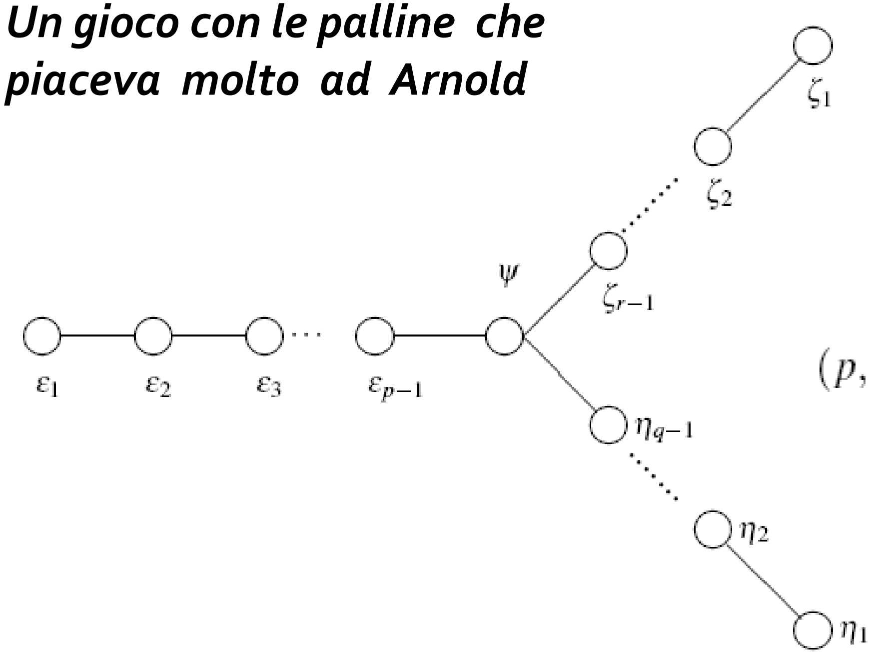
CVM PRIVILEGIO REGIS.

Le equazioni a coefficienti interi le cui incognite devono essere anche risolte con numeri interi si chiamano ***Equazioni Diofantine***

Un gioco con le palline che piaceva molto ad Arnold

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$$

$$p \geq q \geq r$$



Le sole soluzioni sono

$$(p,q,r)=\begin{cases}(\ell,1,1) & \Rightarrow A_\ell\\(\ell-2,2,2) & \Rightarrow D_\ell\\(3,3,2) & \Rightarrow E_6\\(4,3,2) & \Rightarrow E_7\\(5,3,2) & \Rightarrow E_8\end{cases}$$

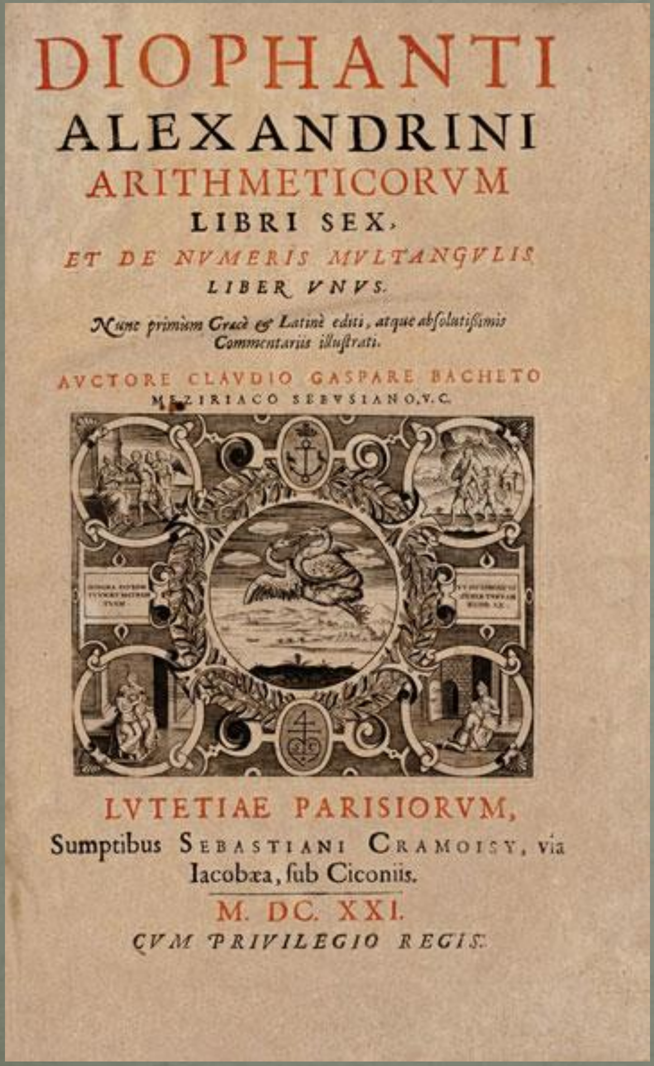
Diophantus Alexandrinus

Diophantus *Arithmetica* (Ἀριθμητικά)

Mount Athos, Pantokrator Monastery
Mikhael Psellos (1017-1078)

Dating Diophantus life was possible through a letter of Psellos discovered by Paul Tannery in Spain in late XIXth century.

The role of the Arabs and of the Byzantines


DIOPHANTI
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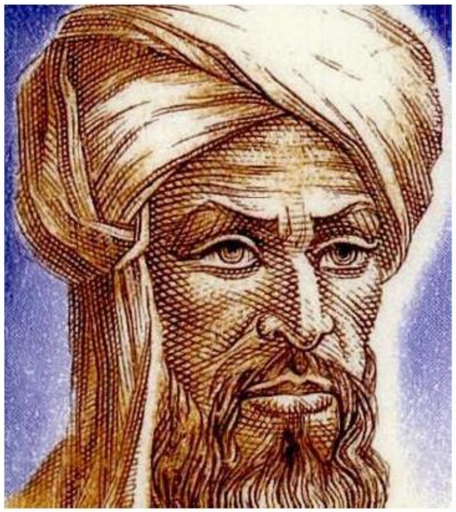
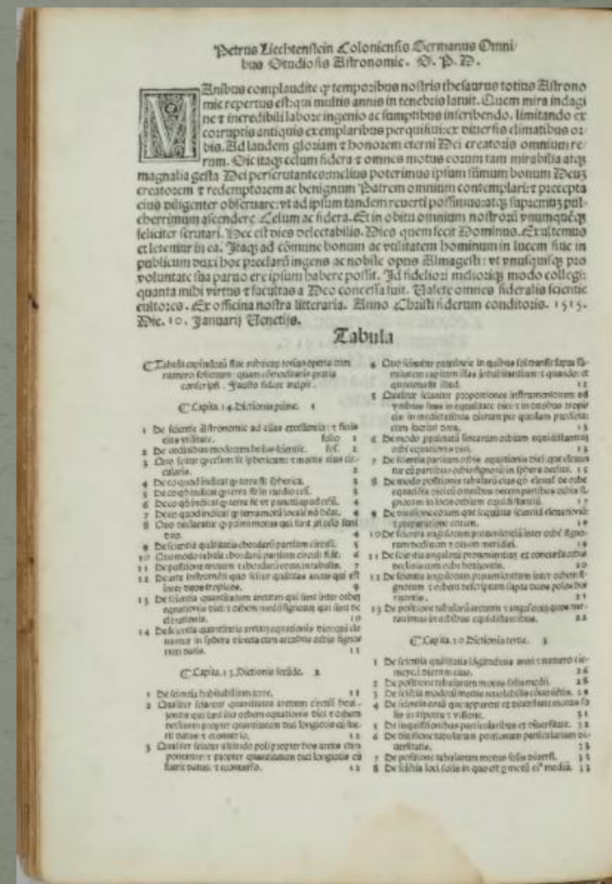
Toledo in the XIIth century A.D. was the meeting place of Arab scholars with Western latin speaking scholars. These latter came here (in a catholic Kingdom) and learned Arabic to talk with their colleagues and retrieve thanks to them the lost Greek science.

Abu Jafar Muhammad ibn Musa al-Khwarizmi lived in Bagdad in the first half of the IXth century. Estimated dates of his birth and death are 780 A.D. and 850 A.D. respectively. Of persian origin he worked at the court of the Abbasid Caliph al-Mamun, who appointed him director of his rich library. Astronomer geographer, mathematician, al-Khwarizmi shares with Diophantus the title of Father of Algebra.





Liber algebrae et almucabala is the title of the translation into latin performed by Robert of Chester and later repeated by Gherardo da Cremona who translated also the Almagest, namely the main book of Ptolemy

The city of Khiva, originally Khwarazm

Some Etimology

kh(w)ar "low" and zam "land." Khwarazm is indeed the lowest region in Central Asia

The name also appears in Achaemenid inscriptions as *Huvarazmish*, which is declared to be part of the Persian Empire.

La città di Khiva è in Uzbekistan e ospita una statua di Al Khwarizmi, cioè quello di Khwarazm, un persiano......arabizzato.

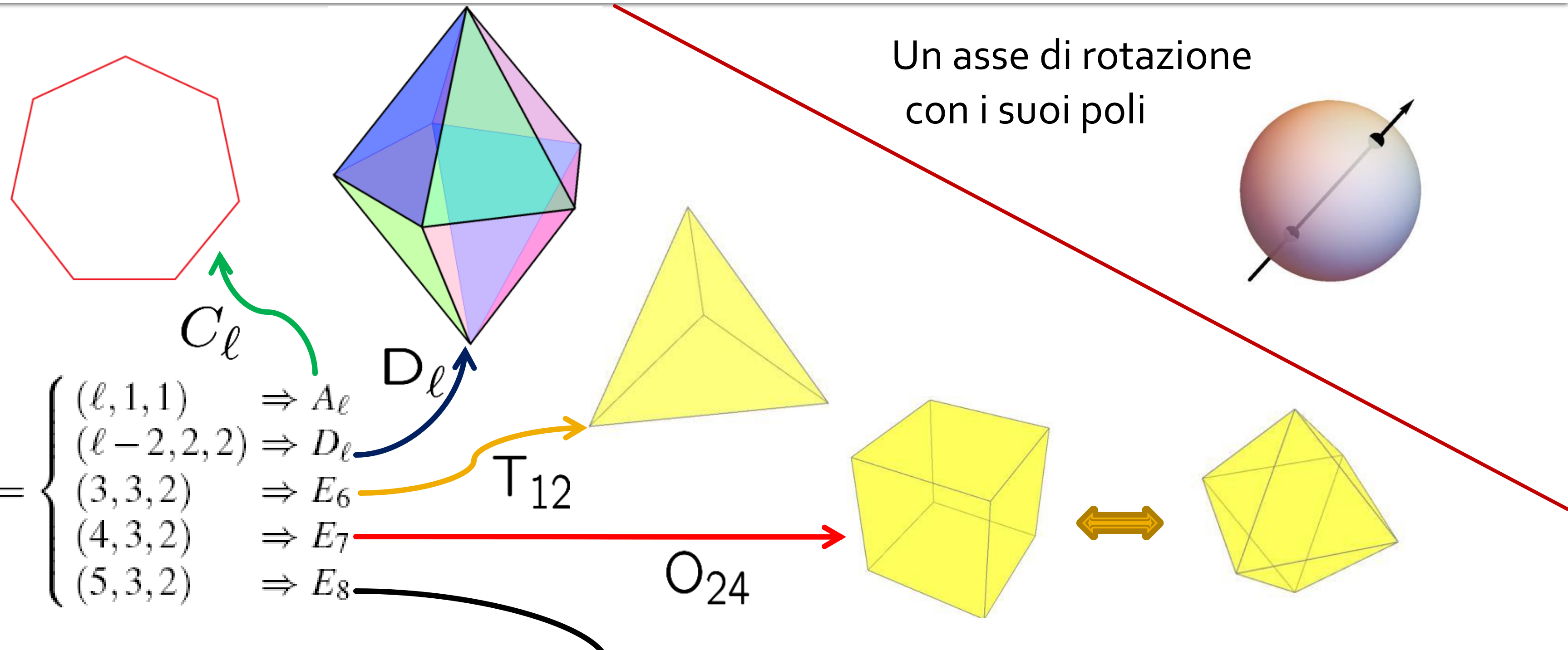
Gerardo da Cremona

Gerardus Cremonensis; (Cremona 1114–Toledo 1187)

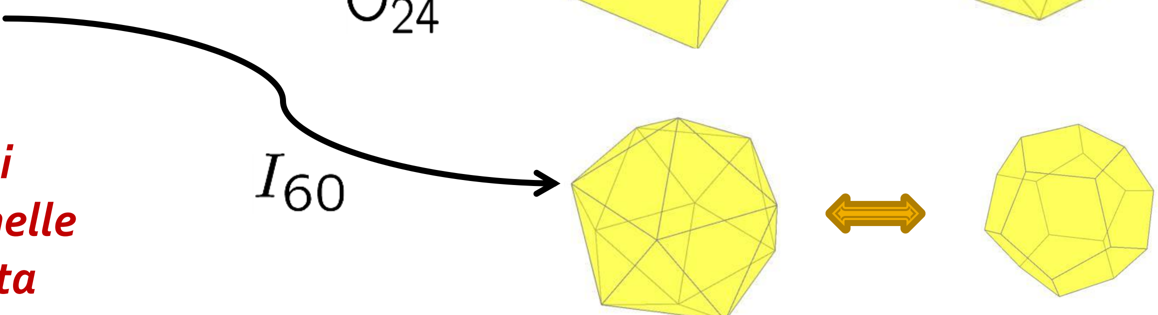
Escuela de Traductores de Toledo
Promoted by King Alphonse the Wise of Castille and Leon (1221-1284)

Gerardo da Cremona lived almost 40 years in Toledo where he studied the arabic language and, always in touch with Arabic and Hebrew Scholars (in particular Rabbi ben Ezra) , pursued a gigantic programme of translation into latin of the Books of Islamic scholars and scientists (about 74 books were translated by him). Several of such books were arabic translations from the Greek of the main sources of Hellenistic Science, like the Ptolomey Almagest, the Elements of Euclid or the surviving books of Diophantus.

Feynmann disse: *le stesse equazioni hanno le stesse soluzioni…*



$$(p,q,r) = \begin{cases} (\ell,1,1) & \Rightarrow A_\ell \\ (\ell-2,2,2) & \Rightarrow D_\ell \\ (3,3,2) & \Rightarrow E_6 \\ (4,3,2) & \Rightarrow E_7 \\ (5,3,2) & \Rightarrow E_8 \end{cases}$$

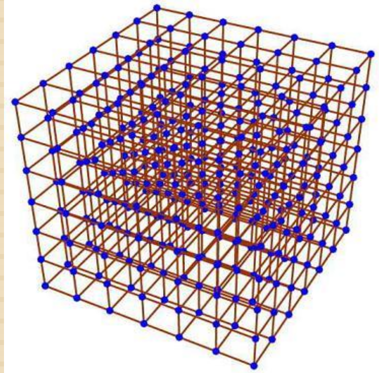


r = numero di orbite di poli
p = frazione di 360 gradi nelle rotazioni della prima orbita
q = frazione di 360 gradi nelle rotazioni della seconda orbita

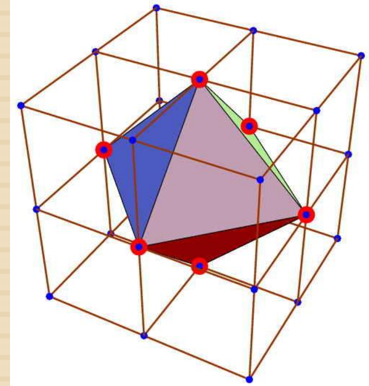
Arnold dimostro' che la stessa equazione diofantina classifica anche le catastrofi

The cubic lattice is self –dual. The rotation subgroup of invariance is the octahedral group O_{24}

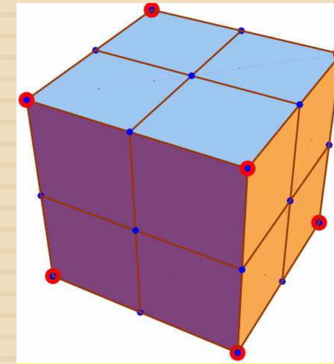
In the momentum lattice there are 4 types of orbits under O_{24}



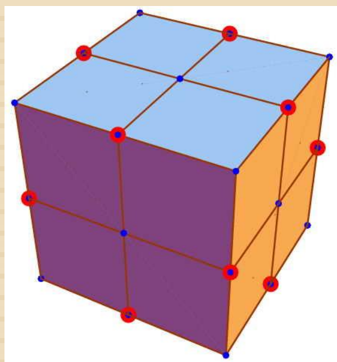
The cubic lattice



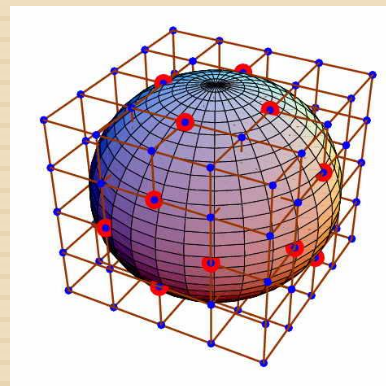
The orbit of legth 6



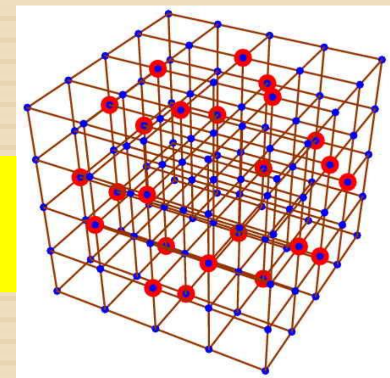
The orbit of legth 8



The orbit of legth 12



The orbit of length 24



Classification of Beltrami fields on T^3

Obtained by Fre & Sorin in 2014-2015

$\mathrm{Y}\,(\mathrm{X}|\mathrm{F})$

To each O_{24}-orbit in momentum lattice of length r we associate a solution of Beltrami equation depending on r parameters **F** and in one case on 2r parameters **F**

The universal classifying group

$$\mathbf{Y}\left(\vec{\mathbf{X}} + \mathfrak{c}|\mathbf{F}\right) = \mathbf{Y}\left(\vec{\mathbf{X}}|\mathcal{M}_{\mathfrak{c}}\mathbf{F}\right)$$

Translations are represented by linear transformations on parameters

$$\forall \gamma \in \mathrm{O}_{24} : \mathbf{Y}\left(\gamma \cdot \vec{\mathbf{X}}\,|\,\mathbf{F}\right) = \mathbf{Y}\left(\vec{\mathbf{X}}\,|\,\mathfrak{R}[\gamma] \cdot \mathbf{F}\right)$$

octahedral rotations are represented by linear transformations of parameters

$$\mathbf{Y}\left(\gamma \cdot \vec{\mathbf{X}} + \mathfrak{c}|\mathbf{F}\right) = \mathbf{Y}\left(\vec{\mathbf{X}}\,|\,\mathfrak{R}[\gamma] \cdot \mathcal{M}_{\mathfrak{c}} \cdot \mathbf{F}\right)$$

We obtain a direct product group

Frobenius congruences

We want to eliminate those roto-translations, that conjugated with translations can be reduced to pure rotations

We are left with a discrete group (containing space groups of crystallography)

$$\mathrm{G}_{1536} \simeq \mathrm{O}_{24} \ltimes (\mathbb{Z}_4 \otimes \mathbb{Z}_4 \otimes \mathbb{Z}_4)$$

Beltrami fields can be classified into irreducible representations of G_{1536}

A few info about G_{1536}

SOLVABLE GROUP

$$G_{1536} \triangleright G_{768} \triangleright G_{256} \triangleright G_{128} \triangleright G_{64}$$

$$\frac{G_{1536}}{G_{768}} \sim \mathbb{Z}_2 \quad ; \quad \frac{G_{768}}{G_{256}} \sim \mathbb{Z}_3 \quad ; \quad \frac{G_{256}}{G_{128}} \sim \mathbb{Z}_2 \quad ; \quad \frac{G_{128}}{G_{64}} \sim \mathbb{Z}_2$$

37 conjugacy classes

1) 2 classes of length 1
2) 2 classes of length 3
3) 2 classes of length 6
4) 1 class of length 8
5) 7 classes of length 12
6) 4 classes of length 24
7) 13 classes of length 48
8) 2 classes of length 96
9) 4 classes of length 128

37 irreducible representations

a) 4 irreps of dimension 1, namely $D_1, \ldots, D_4$

b) 2 irreps of dimension 2, namely $D_5, \ldots, D_6$

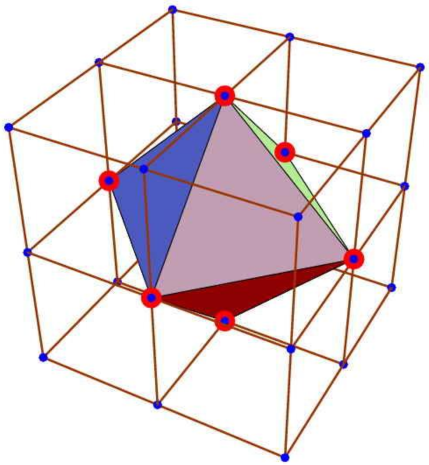
c) 12 irreps of dimension 3, namely $D_6, \ldots, D_{18}$

d) 10 irreps of dimension 6, namely $D_7, \ldots, D_{28}$

e) 3 irreps of dimension 8, namely $D_{29}, \ldots, D_{31}$

f) 6 irreps of dimension 12, namely $D_{32}, \ldots, D_{37}$

The ABC model



$$\mathfrak{W}(\mathbf{X}) = \begin{cases} \mathbf{W}^1 &= 2\,\mathrm{d}X\,\cos(2\pi Z) - 2\,\mathrm{d}Y\,\sin(2\pi Z) \\ \mathbf{W}^2 &= 2\,\mathrm{d}X\,\cos(2\pi Y) + 2\,\mathrm{d}Z\,\sin(2\pi Y) \\ \mathbf{W}^3 &= 2\,\mathrm{d}Y\,\cos(2\pi X) - 2\,\mathrm{d}Z\,\sin(2\pi X) \end{cases}$$

$$G_{bosonic} = \mathrm{O}_{24} \otimes \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

$$\mathrm{O}_{24} = \left(\mathrm{T},\mathrm{S}\,|\,\mathrm{T}^3 = 1,\,\mathrm{S}^2 = 1,\,(\mathrm{S}\,\mathrm{T})^4 = 1\right)$$

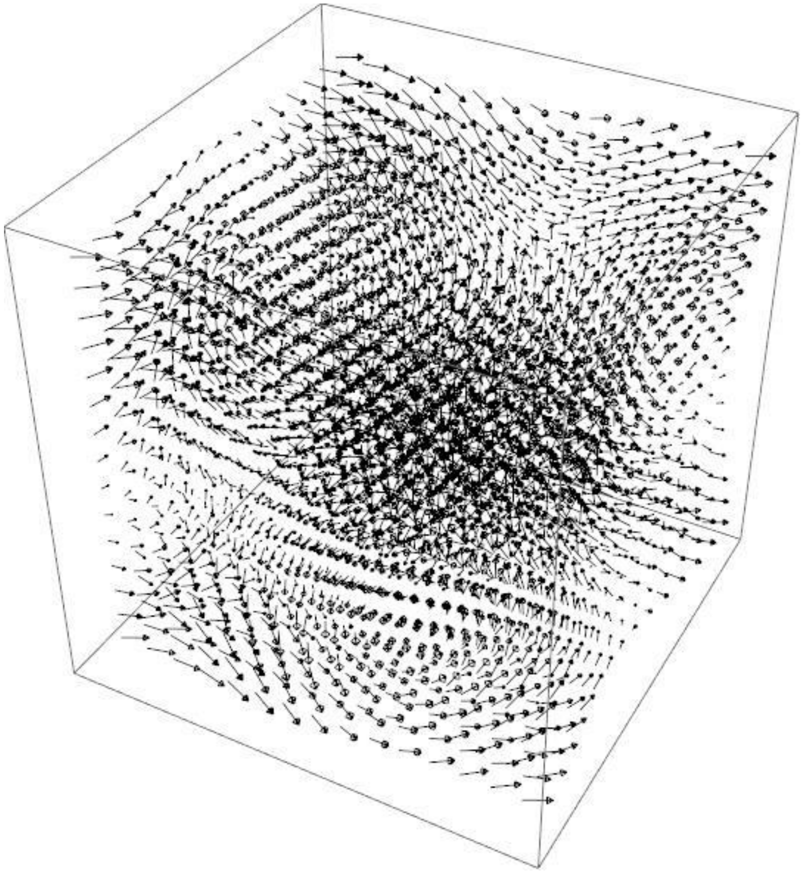
$$\mathfrak{T}\mathbf{X} = \left\{\frac{3}{4} - Y, Z + \frac{1}{4}, -X - \frac{1}{2}\right\} \qquad \mathfrak{W}(\mathfrak{T}\mathbf{X}) = \mathrm{D}[\mathrm{T}]\cdot\mathfrak{W}(\mathbf{X})$$

$$\mathfrak{S}\mathbf{X} = \left\{X + \frac{1}{2}, Y + \frac{1}{2}, Z + \frac{1}{2}\right\} \qquad \mathfrak{W}(\mathfrak{S}\mathbf{X}) = \mathrm{D}[\mathrm{S}]\cdot\mathfrak{W}(\mathbf{X})$$

$$\mathfrak{Z}\mathbf{X} = \left\{\frac{1}{2} - Y, X, Z - \frac{3}{4}\right\} \qquad \mathfrak{W}(\mathfrak{Z}\mathbf{X}) = \mathrm{D}[\mathrm{Z}]\cdot\mathfrak{W}(\mathbf{X})$$

$$\mathrm{D}[\mathrm{T}] = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \quad ; \quad \mathrm{D}[\mathrm{S}] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad ; \quad \mathrm{D}[\mathrm{Z}] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The ABC flow with $\Gamma = O_{24} \otimes \mathbb{Z}_2$



$$H(y) = 1 - \frac{1}{8}\lambda^2 e^{4\pi U}$$

A sentimental journey from…

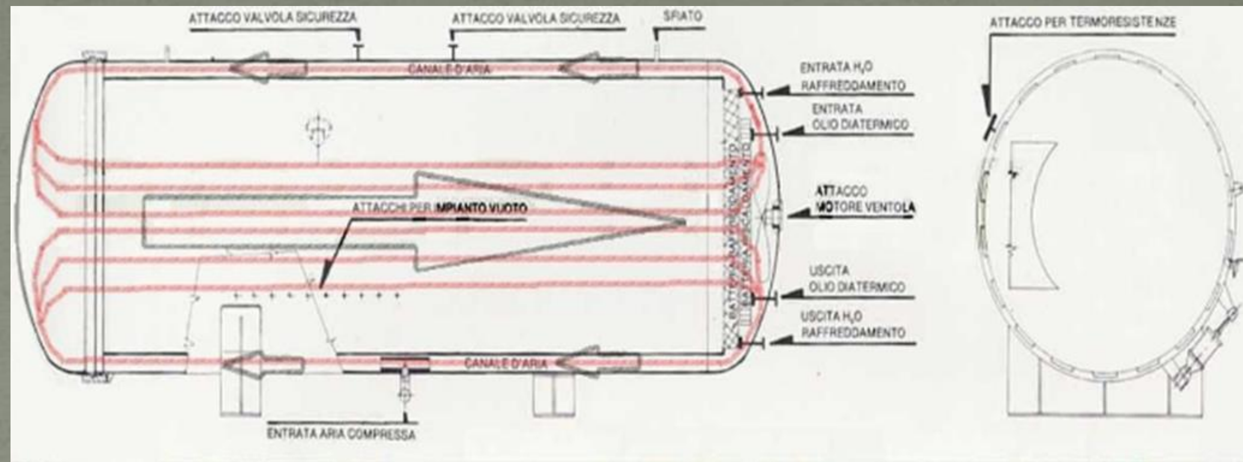
Hydrodynamics to Supergravity

Could we use Beltrami Fields as Gauge Fields in some Supergravity Exact Solution interpreting them as Fluxes,
rather than Flows?

Yes we can! We need 7-dimensional supergravity and we have to look at 2-brane like solutions!

This way of thinking eventually leads us to M-theory and to discover that Beltrami equation is just a subcase of Englert Equation

The autoclav for the curing of composite materials



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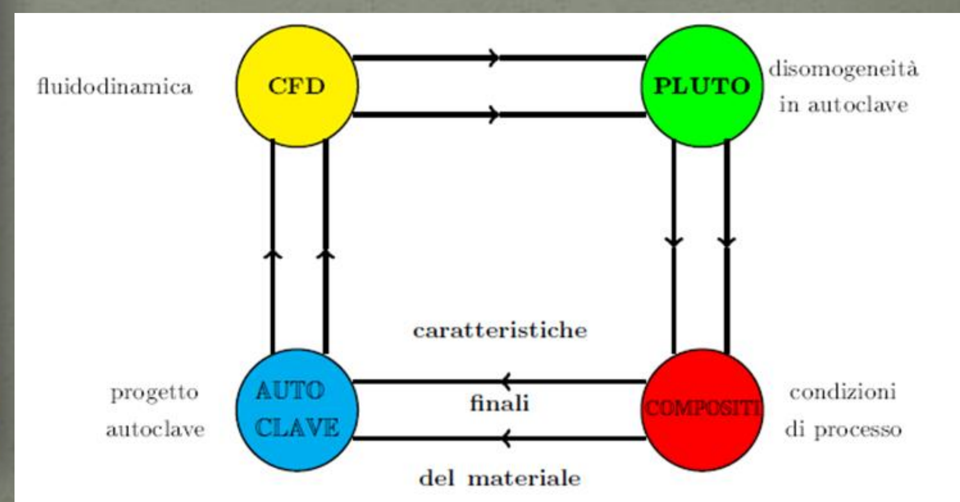
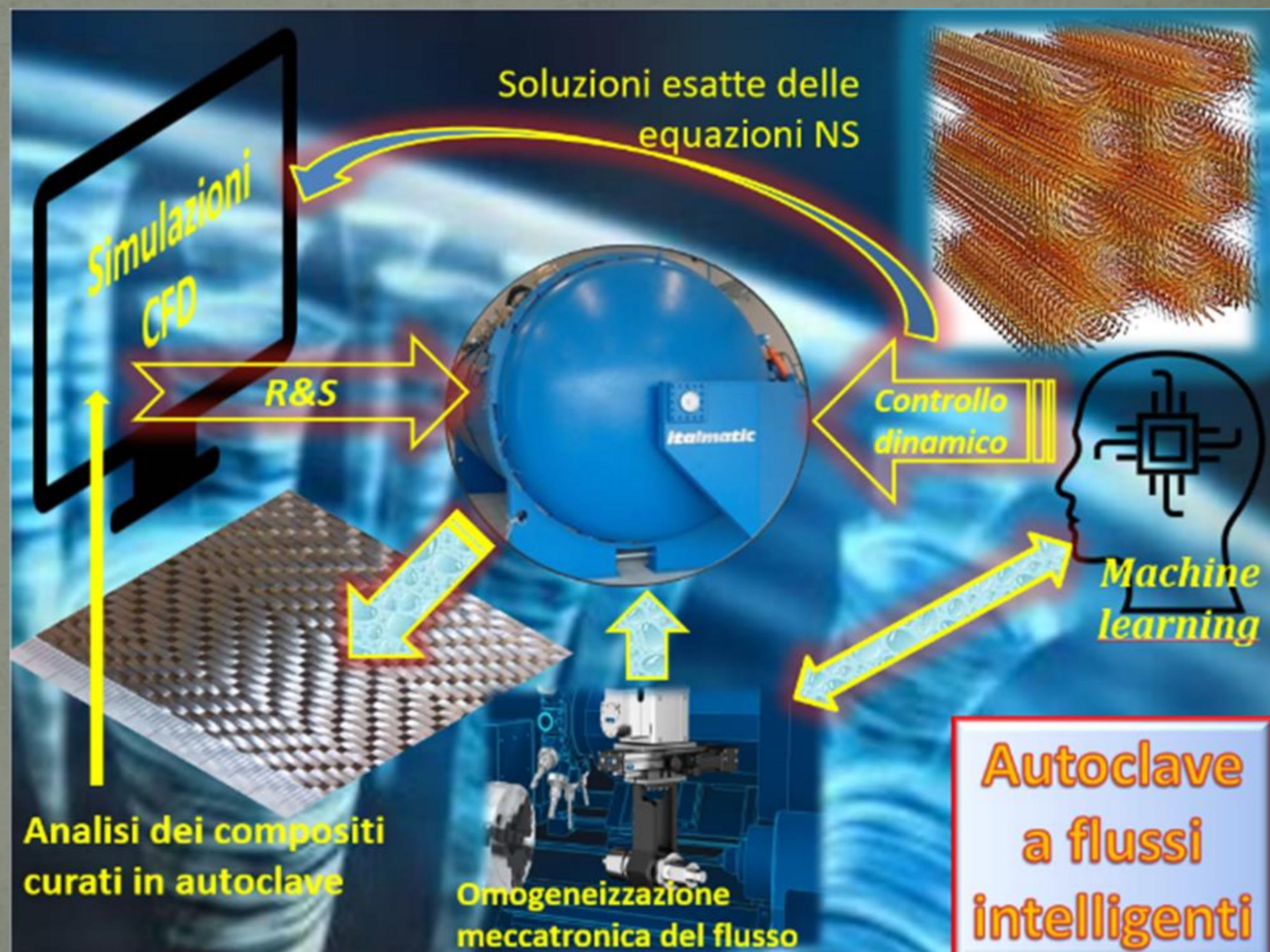


The project of the new autoclav

ALMA
FLUIDA
of
ITALMATIC
PRESSE &
STAMPI

Consulenza Politecnico di Torino



Mathematical Model of the Autoclav