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Extremal Black Holes, Nilpotent Orbits and Tits Satake Universality classes

Based on work in collaboration with A. Sorin

D=4 Supergravity

$$\mathcal{L}^{(4)} = \sqrt{|\det g|}\left[\frac{R[g]}{2} - \frac{1}{4}\partial_\mu\phi^a\partial^\mu\phi^b h_{ab}(\phi) + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu}F^{\Sigma|\mu\nu}\right] + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma} F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} ,$$

$$\mathcal{M}^{D=4}_{scalar} = \frac{\mathrm{U}_{\mathrm{D=4}}}{\mathrm{H}_c}$$

Under rotating extremal black holes

$$ds^2 = -\exp[U]\left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\, dx^i \otimes dx^j\, \delta_{ij}$$

Generic Static Solutions

$$ds^2 = -\exp[U]\left(dt + \mathbf{A}^{[KK]}\right)^2 + \exp[-U]\, dx^i \otimes dx^j\, \gamma_{ij}(x)$$

Sigma model representation of static solutions

$$\mathcal{A}^{[3]} = \int \sqrt{\det\gamma}\,\Re[\gamma]\,d^3x + \int \sqrt{\det\gamma}\,\mathcal{L}^{(3)}\,d^3x$$

$$\mathcal{L}^{(3)} = \Big(\partial_i U\,\partial_j U + h_{rs}\,\partial_i\phi^r\,\partial_j\phi^s + e^{-2U}\left(\partial_i a + \mathbf{Z}^T\mathbb{C}\partial_i\mathbf{Z}\right)\left(\partial_j a + \mathbf{Z}^T\mathbb{C}\partial_j\mathbf{Z}\right) + 2\,e^{-U}\,\partial_i\mathbf{Z}^T\,\mathcal{M}_4\,\partial_j\mathbf{Z}\Big)\,\gamma^{ij}$$

D=3 scalars

	Generic		$\mathcal{N}=2$
warp factor	$U(x)$	1	1
Taub Nut field	$a(x)$	1	1
D=4 scalars	$\phi^a(x)$	n_s	$2n$
Scalars from vectors	$Z^M(x) = \left(Z^\Lambda(x)\,,\,Z_\Sigma(x)\right)$	$2n_V$	$2n+2$
Total		$2+n_s+2n_V$	$4n+4$

$$\mathcal{M}_4 = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \\ \hline \mathrm{Re}\mathcal{N}\,\mathrm{Im}\,\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\,\mathrm{Re}\mathcal{N} \end{array}\right)$$

New coset manifold with Lorentzian signature

$$\mathcal{Q} = \frac{\mathrm{U}_{D=3}}{\mathrm{H}^\star}$$

Oxidation

$$
\begin{aligned}
\mathbf{F}^{[KK]} &= d\mathbf{A}^{[KK]} \\
\mathbf{F}^{[KK]} &= -\epsilon_{ijk}\, dx^i \wedge dx^j \left[\exp[-2U]\left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right)\right] \\
\mathbf{F}^{\Lambda} &= \mathbb{C}^{\Lambda M}\partial_i Z_M\, dx^i \wedge (dt + \mathbf{A}^{[KK]}) \\
&\quad + \epsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U]\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\left(\partial^k Z_\Sigma + \mathrm{Re}\mathcal{N}_{\Sigma\Gamma}\partial^k Z^\Gamma\right)\right]
\end{aligned}
$$

To any solution of the euclidian sigma model field equations we can associate a static solution of Supergravity and all static solutions are retrieved in this way.

The charges are defined by integration on all non trivial homology two-spheres

$$
\begin{aligned}
\mathcal{Q}_\alpha \equiv \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix}_\alpha &= \frac{1}{4\pi\sqrt{2}} \int_{\mathbb{S}^2_\alpha} \epsilon_{ijk} dx^i \wedge dx^j \left[\exp[-U]\,\mathcal{M}_4\,\partial^k Z \right. \\
&\quad \left. + \exp[-2U]\left(\partial^k a + Z\,\mathbb{C}\,\partial^k Z\right)\mathbb{C}\,Z\right]
\end{aligned}
$$

Already known, Bossard, Nicolai et al

Two Main Results

I. A finite, four step algorithm that constructs the generic supergravity solution associated with every nilpotent orbit:

a) Nilpotent algebra associated with {h,X,Y}
b) Harmonic function parametrization of the coset representative in the symmetric gauge
c) Universal formula for the transition to the triangular gauge
d) Extraction of the supergravity fields from the triangular coset representative.

II. Reduction to the Tits Satake subalgebra and organization of supergravity models in a finite list of TS universality classes.

CRUCIAL INGREDIENT
NEW

Nilpotent orbits & subalgebras

The standard triple

$$[h, X] = 2\,X \quad ; \quad [h, Y] = -2\,Y \quad ; \quad [X, Y] = 2\,h$$

The nilpotent algebra

$$\begin{aligned} [h\,,\,C_\mu] &= \mu\,C_\mu \\ \mathbb{N} &= \text{span}\,[C_2\,,\,C_3\,,\,\dots\,,\,C_{max}] \\ \mathbb{N}_{\mathbb{K}} &\equiv \mathbb{N} \bigcap \mathbb{K}^\star \quad ; \quad \mathbb{U} = \mathbb{H}^\star \oplus \mathbb{K}^\star \end{aligned}$$

Graded decomposition

$$\begin{aligned} \mathbb{N}_{\mathbb{K}} &= \bigoplus_{a=0}^{n} \mathbb{N}_{\mathbb{K}}^{(a)} \\ \mathcal{D}^i\mathbb{N}_{\mathbb{K}} &= \mathbb{N}_{K}^{(i)} \oplus \mathcal{D}^{i+1}\mathbb{N}_{\mathbb{K}} \end{aligned}$$

Symmetric Coset representative

$$\begin{aligned} \mathcal{Y}(x) &= \exp\left[\mathfrak{H}(\vec{x})\right] \\ \mathfrak{H}(\vec{x}) &= \sum_{\alpha=0}^{n} \underbrace{\sum_{i=1}^{\ell_\alpha} \mathfrak{h}_i^{(\alpha)}(\vec{x})\,A_\alpha^i}_{\in \mathbb{N}_{\mathbb{K}}^{(\alpha)}} \end{aligned}$$

Hierarchical integration

$$
\begin{aligned}
\nabla^2 \mathfrak{h}_i^{(0)} &= 0 \\
\nabla^2 \mathfrak{h}_i^{(1)} &= \mathfrak{F}_i^{(1)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}\right) \\
\nabla^2 \mathfrak{h}_i^{(2)} &= \mathfrak{F}_i^{(2)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla \mathfrak{h}^{(1)}\right) \\
\dots &= \dots \\
\nabla^2 \mathfrak{h}_i^{(n)} &= \mathfrak{F}_i^{(n)}\left(\mathfrak{h}^{(0)}, \nabla \mathfrak{h}^{(0)}, \mathfrak{h}^{(1)}, \nabla \mathfrak{h}^{(1)}, \dots, \mathfrak{h}^{(n-1)}, \nabla \mathfrak{h}^{(n-1)}\right),
\end{aligned}
$$

This is the brilliant result of Bossard et al …………

However from the symmmetric coset we cannot extract the supergravity fields

We need to transform the symmetric coset to the TRIANGULAR GAUGE!

General formula for the solvable gauge representative

$$\mathbb{L}(\mathcal{Y})\,\mathcal{Q}(\mathcal{Y}) = \mathcal{Y} \quad ; \quad \mathcal{Q}(\mathcal{Y}) \in \mathsf{H}^\star$$

$$\mathbb{L}(\mathcal{Y}) = \begin{pmatrix} L_{1,1}(\mathcal{Y}) & L_{1,2}(\mathcal{Y}) & \cdots & L_{1,n-1}(\mathcal{Y}) & L_{1,n}(\mathcal{Y}) \\ 0 & L_{2,2}(\mathcal{Y}) & \cdots & L_{2,n-1}(\mathcal{Y}) & L_{2,n}(\mathcal{Y}) \\ 0 & 0 & L_{3,3}(\mathcal{Y}) & \cdots & L_{3,n}(\mathcal{Y}) \\ \vdots & \cdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & L_{3,n}(\mathcal{Y}) \end{pmatrix}$$

Determinants

$$\mathfrak{D}_i(\mathcal{Y}) := \mathrm{Det}\begin{pmatrix} \mathcal{Y}_{1,1} & \cdots & \mathcal{Y}_{1,i} \\ \vdots & \vdots & \vdots \\ \mathcal{Y}_{i,1} & \cdots & \mathcal{Y}_{i,i} \end{pmatrix}, \quad \mathfrak{D}_0(\mathcal{Y}) := 1\,.$$

Matrix elements

$$\left(\mathbb{L}(\mathcal{Y})^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathcal{Y})\mathfrak{D}_{i-1}(\mathcal{Y})}}\mathrm{Det}\begin{pmatrix} \mathcal{Y}_{1,1} & \cdots & \mathcal{Y}_{1,i-1} & \mathcal{Y}_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{Y}_{i,1} & \cdots & \mathcal{Y}_{i,i-1} & \mathcal{Y}_{i,j} \end{pmatrix}$$

Extraction of the supergravity fields

$$\mathbb{L}(\Phi) = \exp\left[-a\,L_{+}^{E}\right]\,\exp\left[\sqrt{2}\,Z^{M}\,\mathcal{W}_{M}\right]\,\mathbb{L}_4(\phi)\,\exp\left[U\,L_{0}^{E}\right]$$

Universal decomposition of the D=3 Lie algebra

$$\mathrm{adj}(\mathbb{U}_{D=3}) = \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,\mathbf{W})}$$

Example of solution in the non BPS orbit o322

$$\exp[-U] = \sqrt{\mathcal{H}_2\mathcal{H}_3^3 - 4a_1^2}$$

$$\mathrm{Im}\, z = \frac{\sqrt{\mathcal{H}_2\mathcal{H}_3^3 - 4a_1^2}}{\mathcal{H}_3^2}$$

$$\mathrm{Re}\, z = -\frac{2a_1}{\mathcal{H}_3^2}$$

$$\mathbf{A}^{[KK]} = 0$$

$$j^{EM} = \star\nabla \begin{pmatrix} 0 \\ -\frac{\mathcal{H}_2}{\sqrt{2}} \\ -\sqrt{\frac{3}{2}}\mathcal{H}_3 \\ 0 \end{pmatrix}$$

$$\nabla^2 \mathcal{H}_{1,2} = 0$$

All the fields are parametrerized in terms of two harmonic functions
Assigning the position of the
poles we have multicenter non spherical symmetric solutions

Structure of the algebras and Tits Satake projection

The non compact rank of a non compact coset is defined as

$$
\begin{aligned}
r_{nc} &= \mathrm{rank}\,(\mathrm{U/H}) \equiv \dim \mathcal{H}^{n.c.} \\
\mathcal{H}^{n.c.} &\equiv \mathrm{CSA}_{\mathbb{U}(\mathbb{C})} \bigcap \mathbb{K}
\end{aligned}
$$

If the Cartan subalgebra contains also compact elements U/H is non maximally split

Tits Satake is a geometrical projection of the Root system on its non compact subspace

$$
\Pi^{TS} \quad ; \quad \Delta_{\mathbb{U}} \mapsto \overline{\Delta}_{\mathbb{U}^{TS}}
$$

Concept of Paint Group (P.F. VanProeyen, Rulik, Trigiante 2007)

$$
\mathbb{G}_{\mathrm{paint}} = \mathrm{Aut}_{\mathrm{Ext}}\left[Solv_{\mathrm{U/H}}\right] = \frac{\mathrm{Aut}\,\left[Solv_{\mathrm{U/H}}\right]}{Solv_{\mathrm{U/H}}}
$$

The Tits Satke projection commutes with the dimensional reduction

$$
\begin{aligned}
\mathrm{adj}(\mathbb{U}_{D=3}) &= \mathrm{adj}(\mathbb{U}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,W)} \\
&\Downarrow \\
\mathrm{adj}(\mathbb{U}^{TS}_{D=3}) &= \mathrm{adj}(\mathbb{U}^{TS}_{D=4}) \oplus \mathrm{adj}(\mathfrak{sl}(2,\mathbb{R})_E) \oplus W_{(2,W^{TS})}
\end{aligned}
$$

Reduction to TS

- A careful group theoretical analysis reveals that in the adjoint rep. of $U_{D=3}$ we have always enough parameters to reduce charges and fields to the Tits Satake subalgebra.
- Essentially we can delete entire representations of the subpaint group.

$$\mathfrak{t} \in \mathbb{G}_{\mathrm{subpaint}} \subset \mathbb{G}_{\mathrm{paint}} \subset \mathbb{U} \quad \Leftrightarrow \quad [\mathfrak{t}, \mathbb{G}_{\mathrm{TS}}] = 0$$