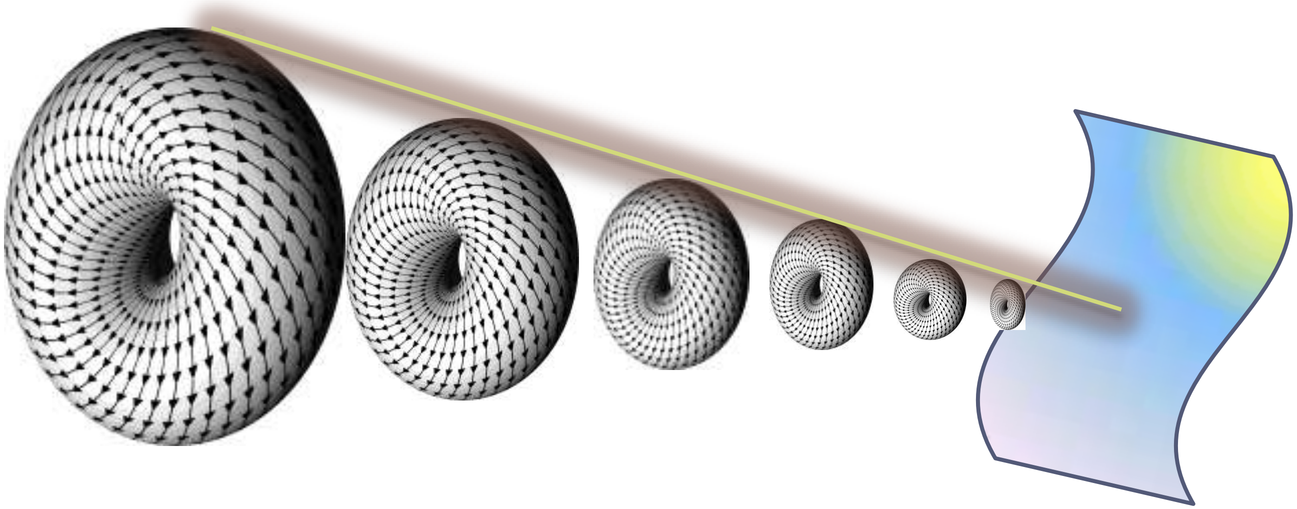


Supergravity & Holography Mini Course @ Viña del Mar

Pietro Fré

LECTURE 1: D=11 SUGRA
Introduction

Constructing D=11 SUGRA, alias M-theory

We start from the multiplet derived in his lectures by Bernard utilizing on-shell state counting

Table 6.2 Structure of the graviton multiplet in $D = 11$ supergravity

SO(1, 10) rep.	# of states	Name
$(2, 0, 0, 0, 0)$	44	graviton
$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	128	gravitino
$(1, 1, 1, 0, 0)$	84	3-form

In the first column of the table we mention the representation of the Lorentz group induced by the helicity representation of SO(9)

There is another way of arriving at the same field content that is more geometrical and not only tells us the states but also the generalized gauge symmetries and provides the tools to obtain all the interactions, namely the field equations.

FREE DIFFERENTIAL ALGEBRAS = needed generalization of the concept of (super) Lie Algebras

Free Differential Algebras

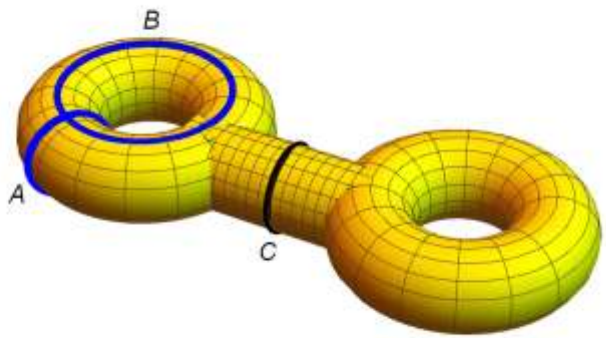
All higher dimensional supergravities and in particular the maximal one in $D = 11$ are based on the gauging of a new type of algebraic structure named *Free Differential Algebras*. *What goes under this name was independently discovered at the* beginning of the eighties in Mathematics by Sullivan and in Physics by R. D'Auria & P.F.

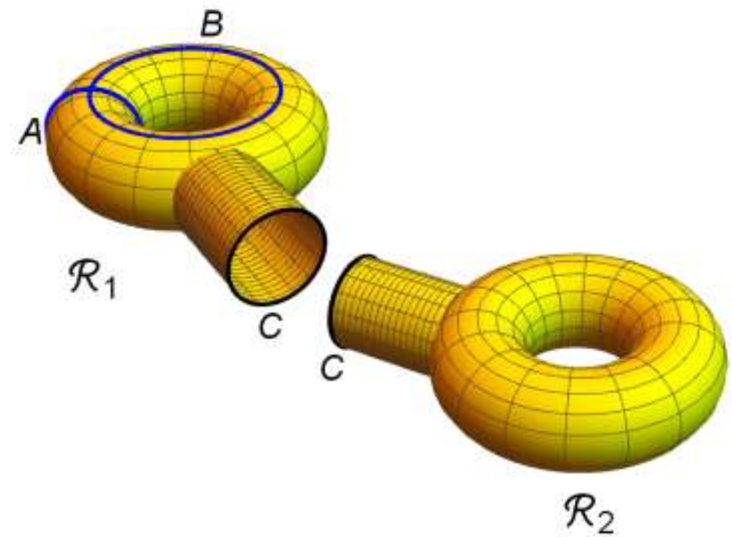
Free Differential Algebras (FDA) are a categorical extension of the notion of Lie algebra and constitute the natural mathematical environment for the description of the algebraic structure of higher dimensional supergravity theory

The reason is the ubiquitous presence in the spectrum of supergravity theories of antisymmetric gauge fields (*p-forms) of rank* greater than one

The very existence of FDAs is a consequence of the *Chevalley cohomology* of ordinary Lie algebras and **Sullivan** has provided us with a very elegant classification scheme of these algebras based on two structural theorems rooted in the set up of such an elliptic complex.

Excursus on homology

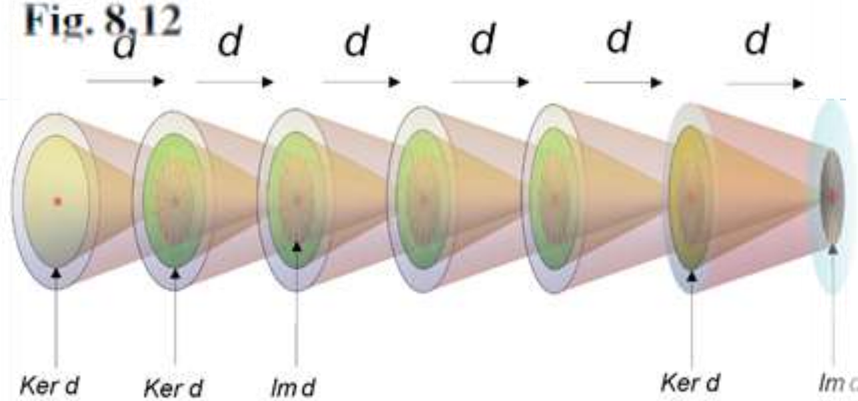




All the curves A, B, C are closedbecausetheyneitherhavea beginning nor an ending. Indeedtheyare loops.Thereisa differencebetweenA,B and C. Ifyoucutthe surfacealongA or B, itdoesnotsplit in twoparts. HenceneitherA, nor B are the boundary of a region. Ifyoucutthe surfacealongC itsplitsin twoparts R_1 and R_2. Hence C is the boundary of thesetworegions. Everyboundaryisclosedbutnotallcurves(or surfaces) are boundaries. THIS IS HOMOLOGY in a nut-shell

Excursus on Cohomology

$$\ker\partial_i \supset \mathrm{Im}\partial_{i-1}$$



Let us begin with Fig. 8.12. The fundamental idea underlying cohomology theory is captured by that image. There is a sequence of *spaces* $\Omega^{[i]}$, whose elements we name the *cochains*[6] and there is a linear operator, named d (the exterior derivative) that provides *non surjective maps* from each space $\Omega^{[i]}$ to the next one $\Omega^{[i+1]}$:

$$\partial_i : \Omega^{[i]} \xrightarrow{d} \Omega^{[i+1]} \quad ; \quad \forall\phi \in \Omega^{[i]} \quad d\phi \in \Omega^{[i+1]}$$

The fundamental property of the operator d is its nilpotency, namely it squares to zero $d^2 = 0$. In practice this means that the kernel of the map ∂_i, whose elements we name the *cocycles*[7] always contains the image $\mathrm{Im}\partial_{i-1}$ of the previous map ∂_{i-1}, namely the subspace of $\Omega^{[i]}$ formed by all those elements that can be written as $d\phi$ for some ϕ belonging to $\Omega^{[i-1]}$. We name *coboundaries* the elements of $\mathrm{Im}\partial_{i-1}$.

Homology or Cohomology classes are equivalence classes

Such a scenario occurs in various mathematical constructions and it is named an *elliptic complex* $\mathscr{C}$. The cohomology groups of the complex, usually denoted $H^{[i]}(\mathscr{C})$ are defined as the set of equivalence classes in which the subspace $\ker\partial_i$ can be partitioned with respect to the following equivalence relation:

$$\forall \omega^{[i]}, \psi^{[i]} \in \ker \partial_i \quad : \quad \omega^{[i]} \sim \psi^{[i]} \quad \text{iff} \quad \left(\omega^{[i]} - \psi^{[i]}\right) \in \mathrm{Im}\partial_{i-1} \qquad (8.2.31)$$

Chevalley Cohomology

Let us consider a (super) Lie algebra $\mathbb{G}$ identified through its structure constants $\tau^{I}{}_{JK}$ which are alternatively introduced through the commutation relation of the generators[7]

$$[T_I, T_K] = \tau^{I}{}_{JK} T_I$$

or the Cartan Maurer equations:

$$\partial e^I = \frac{1}{2} \tau^{I}{}_{JK} e^J \wedge e^K$$

where e^I is an abstract set of left-invariant 1-forms.

The isomorphism between the two descriptions of the Lie algebra is provided by the duality relations:

$$e^I(T_J) = \delta^I_J$$

The Chevalley complex

A p-cochain $\Omega^{[p]}$ of the Chevalley complex is just an exterior p-form on the Lie algebra with constant coefficients, namely:

$$\Omega^{[p]} = \Omega_{I_1 \dots I_p} e^{I_1} \wedge \cdots \wedge e^{I_p}$$

where the antisymmetric tensor $\Omega_{I_1 \dots I_p} \in \bigwedge^p \mathrm{adj}\, \mathbb{G}$, which belongs to the pth antisymmetric power of the adjoint representation of $\mathbb{G}$, has constant components. Using the Maurer Cartan equations (6.3.2) the coboundary operator ∂ has a pure algebraic action on the Chevalley cochains:

$$\partial \Omega^{[p]} = \partial \Omega_{I_1 \dots I_{p+1}} e^{I_1} \wedge \cdots \wedge e^{I_{p+1}}$$

$$\partial \Omega_{I_1 \dots I_{p+1}} = (-)^{p-1} \frac{p}{2} \tau^R{}_{[I_1 I_2} \Omega_{I_1 \dots I_{p+1}]R}$$

and Jacobi identities guarantee the nilpotency of this operation $\partial^2 = 0$.

The cohomology groups $H^{[p]}(\mathbb{G})$ are constructed in standard way. The p-cocycles $\Omega^{[p]}$ are the closed forms $\partial\Omega^{[p]} = 0$ while the exact p-forms, or p-coboundaries, are those $\Lambda^{[p]}$ such that they can be written as $\Lambda^{[p]} = \partial\Phi^{[p-1]}$ for some suitable $(p-1)$-forms $\Phi^{[p-1]}$. The pth cohomology groups is spanned by the p-cocycles modulo the p-coboundaries. Calling $C^p(\mathbb{G})$ the linear space of p-chains the operator ∂ defined in (6.3.5) induces a sequence of linear maps ∂_p:

$$C^0(\mathbb{G}) \xrightarrow{\partial_0} C^1(\mathbb{G}) \xrightarrow{\partial_1} C^2(\mathbb{G}) \xrightarrow{\partial_2} C^3(\mathbb{G}) \xrightarrow{\partial_3} C^4(\mathbb{G}) \xrightarrow{\partial_4} \cdots \tag{6.3.6}$$

and we can summarize the definition of the Chevalley cohomology groups in the standard form used for all elliptic complexes:

$$H^{(p)}(\mathbb{G}) \equiv \frac{\ker \partial_p}{\operatorname{Im} \partial_{p-1}} \tag{6.3.7}$$

General Structure of FDAs and Sullivan's Theorems

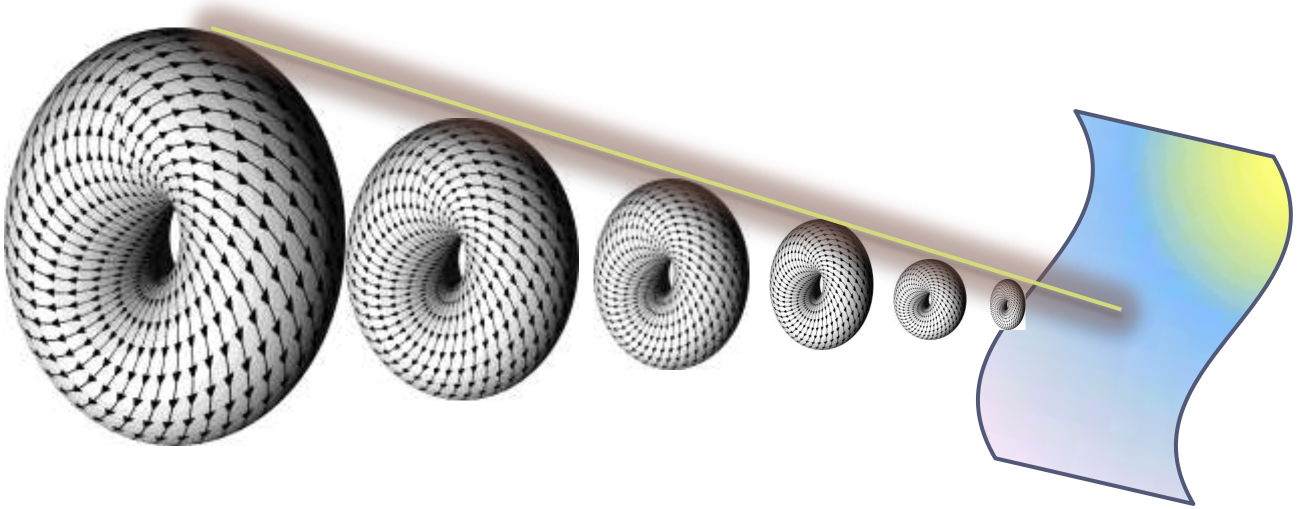
Consider a formal set of exterior forms $\{\theta^{A(p)}\}$ *labeled by the index A and by the degree p, which* may be different for different values of *A*. *Given this set of p-forms we can write* the corresponding set of generalized *Maurer Cartan equations as follows:*

$$d\theta^{A(p)} + \sum_{n=1}^{N} C^{A(p)}{}_{B_1(p_1)\dots B_n(p_n)} \theta^{B_1(p_1)} \wedge \cdots \wedge \theta^{B_n(p_n)} = 0$$

where $C^{A(p)}{}_{B_1(p_1)\dots B_n(p_n)}$ are generalized structure constants with the same symmetry as induced by permuting the θs in the wedge product. They can be non-zero only if:

$$p + 1 = \sum_{i=1}^{n} p_i \qquad (6.3.16)$$

The generalized Maurer Cartan equations are consistent if and only if $d\,d\theta^{A(p)} = 0$ is satisfied. These are the generalized Jacobi identities.



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LECTURE II: D=11 SUGRA Continuation

Classification of FDA and the Analogue of Levi Theorem: Minimal Versus Contractible Algebras

A minimal FDA is defined by

$$C^{A(p)}{}_{B(p+1)} = 0$$

This excludes the case where a *(p + 1)-form* appears in the generalized Maurer Cartan equations as a contribution to the derivative of a *p-form. In a minimal algebra* all non-differential terms are products of at least two elements of the algebra

On the other hand a *contractible FDA* is one where the only form appearing in the expansion of $d\theta^{A(p)}$ has degree $p+1$, namely:

$$d\theta^{A(p)} = \theta^{A(p+1)} \quad \Rightarrow \quad d\theta^{A(p+1)} = 0$$

A contractible algebra has a trivial structure. The basis $\{\theta^{A(p)}\}$ can be subdivided in two subsets $\{\Lambda^{A(p)}\}$ and $\{\Omega^{B(p+1)}\}$ where A spans a subset of the values taken by B, so that:

$$d\Omega^{B(p+1)} = 0$$

for all values of B and

$$d\Lambda^{A(p)} = \Omega^{A(p+1)}$$

Denoting by $\mathbb{M}^k$ the vector space generated by all forms of degree $p \leq k$ and C^k the vector space of forms of degree k, a minimal algebra is shortly defined by the property:

$$\boxed{d\mathbb{M}^k \subset \mathbb{M}^k \wedge \mathbb{M}^k}$$

while a contractible algebra is defined by the property

$$dC^k \subset C^{k+1}$$

In analogy to Levi's theorem, the first theorem by Sullivan states that: *The most general FDA is the semidirect sum of a contractible algebra with a minimal algebra.*

31 years ago

Sullivan's First Theorem and the Gauging of FDAs Twenty five years ago in [16] the present author observed that the above mathematical theorem has a deep physical meaning relative to the gauging of algebras. Indeed he proposed the following identifications:

1. The *contractible generators* $\Omega^{A(p+1)} + \cdots$ of any given FDA $\mathbb{A}$ are to be physically identified with the *curvatures*.
2. The Maurer Cartan equations that begin with $d\Omega^{A(p+1)}$ are *the Bianchi identities*.
3. The algebra which is gauged is the *minimal subalgebra* $\mathbb{M} \subset \mathbb{A}$.
4. The Maurer Cartan equations of the minimal subalgebra $\mathbb{M}$ are consistently obtained by those of $\mathbb{A}$ by setting all contractible generators to zero.

Sullivan's Second Structural Theorem and Chevalley Cohomology

The second structural theorem proved by Sullivan deals with the structure of minimal algebras and it is constructive.

The most general minimal FDA M necessarily contains an ordinary Lie subalgebra G ⊂ M whose associated one-form generators we can call e^I

$$\partial e^I = \frac{1}{2}\tau^I{}_{JK}\, e^J \wedge e^K$$

Additional p-form generators $A^{[p]}$ of $\mathbb{M}$ are necessarily, according to Sullivan's theorem, in one-to-one correspondence with Chevalley $p+1$ cohomology classes $\Gamma^{[p+1]}(e)$ of $\mathbb{G} \subset \mathbb{M}$. Indeed, given such a class, which is a polynomial in the e^I generators, we can consistently write the new higher degree Maurer Cartan equation:

$$\partial A^{[p]} + \Gamma^{[p+1]}(e) = 0 \qquad (6.3.23)$$

where $A^{[p]}$ is a new object that cannot be written as a polynomial in the old objects e^I.

An iterative process

Considering now the FDA generated by the inclusion of the available $A^{[p]}$, one can inspect its Chevalley cohomology: the cochains are the polynomials in the extended set of forms $\{A, e^I\}$ and the boundary operator is defined by the enlarged set of Maurer Cartan equations. If there are new cohomology classes $\Gamma^{[p+1]}(e, A)$, then one can further extend the FDA by including new p-generators $B^{[p]}$ obeying the Maurer Cartan equation:

$$\partial B^{[p]} + \Gamma^{[p+1]}(e, A) = 0$$

The iterative procedure can now be continued by inspecting the cohomology classes of type $\Gamma^{[p+1]}(e, A, B)$ which lead to new generators $C^{[p]}$ and so on. Sullivan's theorem states that those constructed in this way are, up to isomorphisms, the most general minimal FDAs.

Relative Cohomology

To be precise, this is not the whole story. There is actually one generalization that should be taken into account. Instead of *absolute Chevalley cohomology* one can rather consider *relative Chevalley cohomology*. This means that rather then being $\mathbb{G}$- singlets, the Chevalley p-cochains can be assigned to some linear representation of the Lie algebra $\mathbb{G}$. In this case (6.3.4) is replaced by:

$$\Omega^{\alpha[p]} = \Omega^{\alpha}_{I_1 \ldots I_p} e^{I_1} \wedge \cdots \wedge e^{I_p}$$

where the index α runs in some representation D:

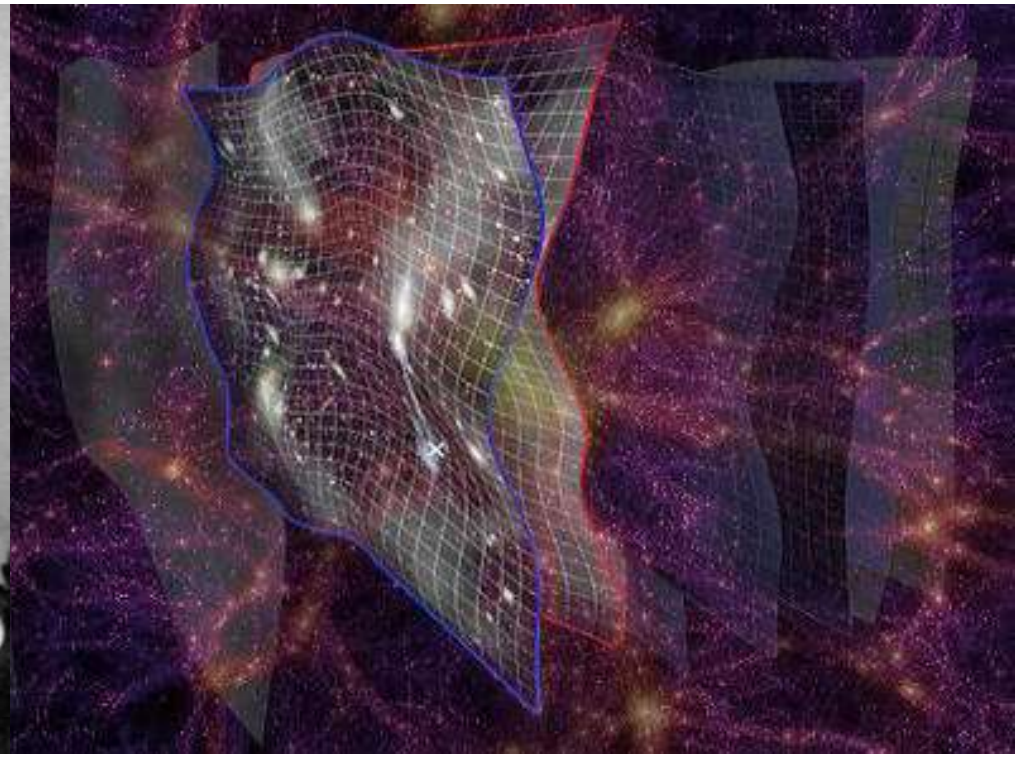
$$D : T_I \to \left[D(T_I)\right]^{\alpha}{}_{\beta}$$

and the boundary operator is now the covariant ∇:

$$\nabla \Omega^{\alpha[p]} \equiv \partial \Omega^{\alpha[p]} + e^I \wedge \left[D(T_I)\right]^{\alpha}{}_{\beta} \Omega^{\beta[p]}$$

Since $\nabla^2 = 0$, we can repeat all previously explained steps and compute cohomology groups. Each non-trivial cohomology class $\Gamma^{\alpha[p+1]}(e)$ leads to new p-form generators $A^{\alpha[p]}$ which are assigned to the same $\mathbb{G}$-representation as $\Gamma^{\alpha[p+1]}(e)$. All successive steps go through in the same way as before and Sullivan's theorem actually states that all minimal FDAs are obtained in this way for suitable choices of the representation D, in particular the singlet.

The Super FDA of M Theory and Its Cohomological Structure



The existence of the 3-form and 6-form gauge field and hence of M2 and M5 branes that couple to them is a cohomological yield of the SuperPoincaré Lie Algebra in D=11. It is the same for the other maximal supergravities in D=10

p-form content of the M-theory FDA

1. the vielbein V^a,
2. the spin connection ω^{ab},
3. the gravitino ψ.

The generators e^I

The higher degree generators of the minimal FDA $\mathbb{M}$ are:

1. the bosonic 3-form $\mathbf{A}^{[3]}$,
2. the bosonic 6-form $\mathbf{A}^{[6]}$.

The new generators

The complete FDA

$$\mathfrak{T}^a = \mathscr{D}V^a - \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma^a \psi$$

$$\mathfrak{R}^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{cb}$$

$$\rho = \mathscr{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\psi$$

Poincaré super Lie Algebra curvatures

$$\mathbf{F}^{[4]} = d\mathbf{A}^{[3]} - \frac{1}{2}\overline{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b$$

$$\mathbf{F}^{[7]} = d\mathbf{A}^{[6]} - 15\mathbf{F}^{[4]} \wedge \mathbf{A}^{[3]} - \frac{15}{2}V^a \wedge V^b \wedge \overline{\psi} \wedge \Gamma_{ab}\psi \wedge \mathbf{A}^{[3]}$$
$$- \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma_{a_1 \dots a_5}\psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5}$$

The new generators

The Minimal FDA of M-Theory and Cohomology

Setting $\mathfrak{T}^a = \mathfrak{R}^{ab} = \rho = \mathbf{F}^{[4]} = \mathbf{F}^{[7]} = 0$ in (6.4.2) we obtain the Maurer Cartan equations of the minimal algebra $\mathbb{M}$. In particular we have:

$$d\mathbf{A}^{[3]} = \Gamma^{[4]}(V,\psi) \equiv \frac{1}{2}\overline{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b$$

$$d\mathbf{A}^{[6]} = \Gamma^{[7]}\big(V,\psi,\mathbf{A}^{[3]}\big)$$

$$\equiv \frac{15}{2} V^a \wedge V^b \wedge \overline{\psi} \wedge \Gamma_{ab}\psi \wedge \mathbf{A}^{[3]}$$

$$+ \mathrm{i}\frac{1}{2}\overline{\psi} \wedge \Gamma_{a_1 \ldots a_5}\psi \wedge V^{a_1} \wedge \cdots \wedge V^{a_5}$$

The reason why the three-form generator $\mathbf{A}^{[3]}$ does exist and also why the six-form generator $\mathbf{A}^{[6]}$ can be included is, in this set up, a direct consequence of the cohomology of the super Poncaré algebra in $D = 11$, via Sullivan's second theorem.

Indeed the 4-form $\Gamma^{[4]}(V,\psi)$ defined in the first line of (6.4.10) is a cohomology class of the super Poincaré Lie algebra whose Maurer Cartan equations are the first three of (6.4.2) upon setting $\mathfrak{T}^a = \mathfrak{R}^{ab} = \rho = 0$. We have:

$$d\Gamma^{[4]}(V,\psi) = 0$$

and there is no $\Phi^{[3]}(V,\psi)$ such that $\Gamma^{[4]}(V,\psi) = d\Phi^{[3]}(V,\psi)$.

The algebraic reason why $\Gamma^{[4]}(V,\psi)$ is a closed form is also rooted in Lie algebra theory and can be expressed in intrinsic group-theoretical terms. It follows from the following Fierz identity:

$$\overline{\psi} \wedge \Gamma^{ab}\psi \wedge \overline{\psi} \wedge \Gamma_a \psi = 0$$

The left hand side is a projection operator on the **11** *irrep*[9] out of the symmetric product of four *irreps* **32**. The reason why the result is zero is that in the Clebsch Gordan expansion of such a four product the *irrep* **11** is not contained. Indeed we have:

$$(\mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32})_{\text{symm}} = \mathbf{1} \oplus \mathbf{165} \oplus \mathbf{330} \oplus \mathbf{462} \oplus \mathbf{65} \oplus \mathbf{429} \oplus \mathbf{4290} \oplus \mathbf{1144} \oplus \mathbf{17160} \oplus \mathbf{32604}$$

The Principle of Rheonomy

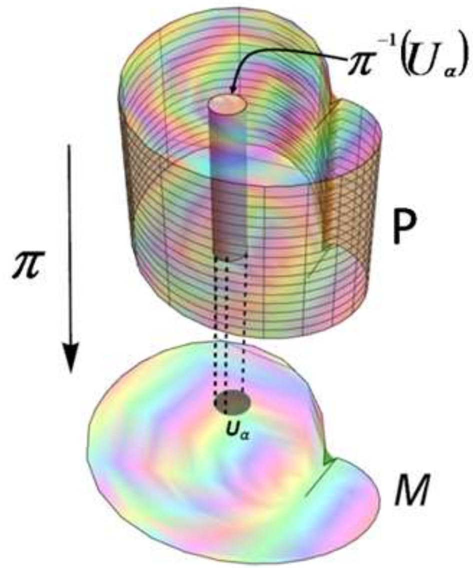
The principle of rheonomy was introduced by D'Auria and P.F. in a paper of 1979 [22], formalizing a previous idea of Ne'eman and Regge [23]

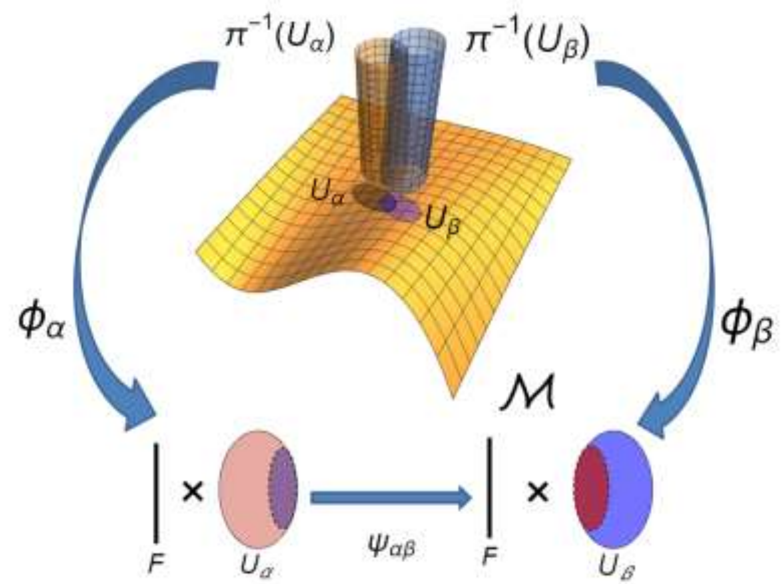
D'Auria, R., Frè, P.: About bosonic rheonomic symmetry and the generation of a spin 1 field in $D = 5$ supergravity. Nucl. Phys. B **173**, 456 (1980)
Ne'eman, Y., Regge, T.: Gravity and supergravity as gauge theories on a group manifold. Phys. Lett. B **74**, 54 (1978)

The basic motivation to introduce such a concept was the geometrical interpretation of *local supersymmetry transformations* at the basis of the newly found theory of supergravity, which, at that time, was less than two year old. In this respect the key problem is that supersymmetry transformations, as they were case by case found in the early construction of supersymmetric theories, look similar to *gauge-transformations*, yet their gauge-field ψ_μ, which ultimately encodes the spin $\frac{3}{2}$ particles, has not a horizontal field-strength and therefore is *not* a proper *connection* on a *principal fibre-bundle*.

In its *strong formulation,* used so far, horizontality requires that the components of the curvatures should be zero in the vertical directions. A *weaker formulation of the same idea is* easily deemed of: one could just require that the vertical components should just be dependent on the horizontal ones, in particular *linear combinations of the latter.*
This very simple idea is the *principle of rheonomy.*

Fibre Bundles in a few pictures





$$\phi_\alpha \;:\; \pi^{-1}(U_\alpha) \subset P \to U_\alpha \otimes F \qquad \pi \circ \phi_\alpha^{-1}(p, f) = p$$

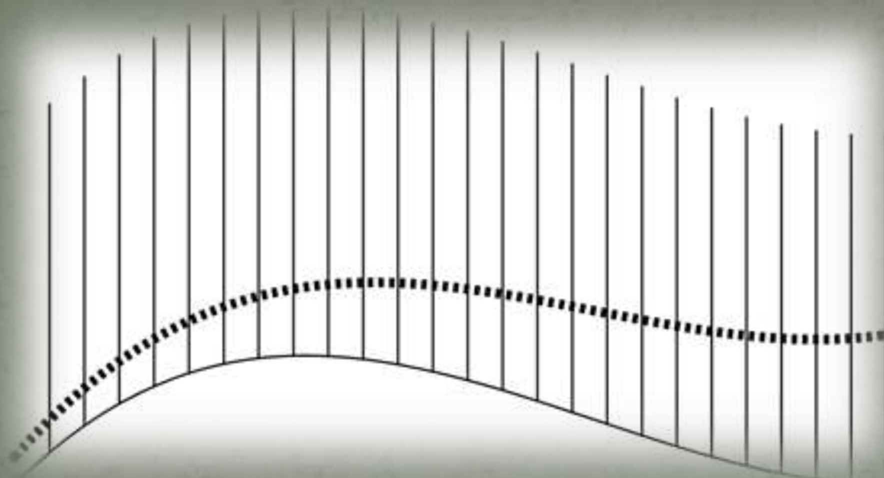
$$t_{\alpha\beta} \equiv \phi_\beta^{-1} \circ \phi_\alpha \;:\; \left(U_\alpha \bigcap U_\beta\right) \otimes F \to \left(U_\alpha \bigcap U_\beta\right) \otimes F$$

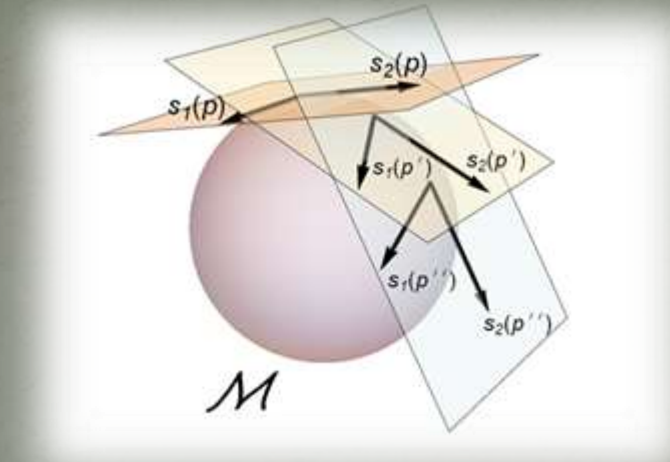
Transition functions

The transition functions……..

- Belong to a Lie Group G (the structural group)
- G acts as a group of transformations on the standard fibre F
- When F=G is the Lie group itself we have a Principal Bundle
- Principal Bundles are the ancestors of an infinite tower of associated vector bundles, one for each linear representation of G.

Section of a bundle





The concept of section of a fibre-bundle is illustrated by the above picture. To every point p of the base manifold a section $\mathfrak{s}$ associates, in a continuous way, a point of the total space $\mathfrak{s}(p) \in P$, that must belong to the fibre over p, namely such that $\pi\,(\mathfrak{s}\,(p)) = p$. In the case of vector bundles the section image $\mathfrak{s}\,(p)$ of a base manifold point p is necessarily an r-dimensional vector, r being the rank of the bundle

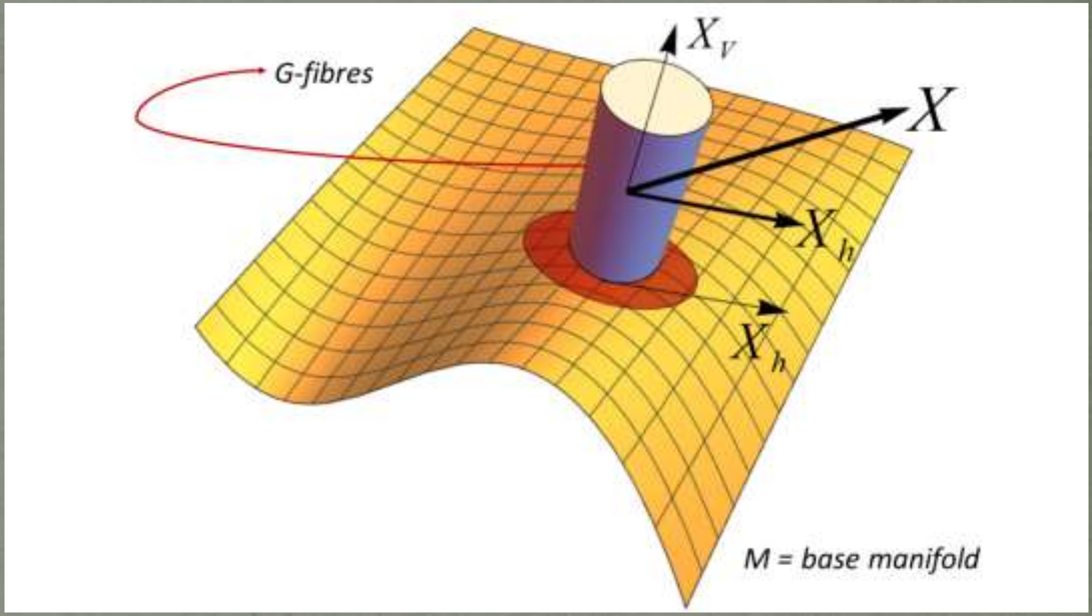
A frame over U is a set of r sections $\{\mathfrak{s}_1, \ldots, \mathfrak{s}_r\}$ such that $\{\mathfrak{s}_1(z), \ldots, \mathfrak{s}_r(z)\}$ is a basis for $\pi^{-1}(p)$ for any $p \in U$, having denoted by z^i the coordinates labeling the points of the base manifold in the chosen patch

In Physics, all matter fields are sections of a vector bundle associated to a Principal Bundle

Ehresman's Connection

Let *P(M, G)* be a principal fibre-bundle. A *connection* on *P* is a rule which at any point $u \in P$ defines a unique splitting of the tangent space $T_u P$ into the vertical subspace $V_u P$ and into a horizontal complement $H_u P$

The algorithmic way to implement the splitting rule advocated by the Ehresmann definition is provided by introducing a connection one-form **A** which is just a Lie algebra valued differential one-form on the bundle *P*

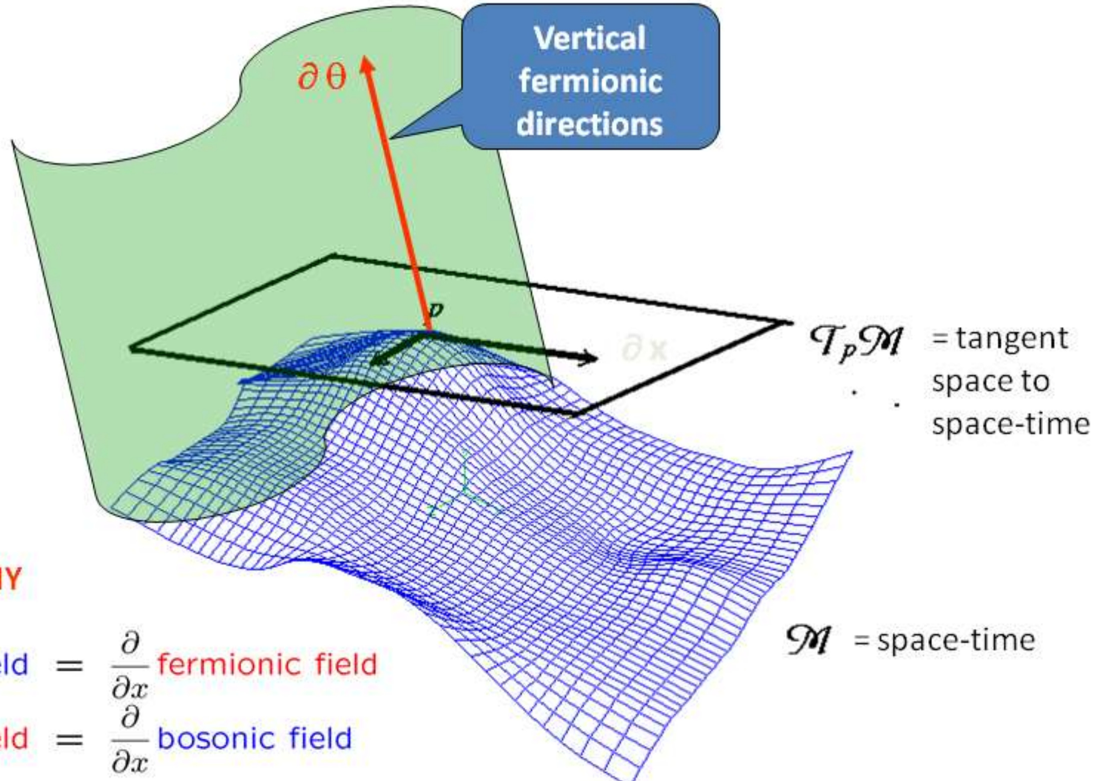


$$
\begin{aligned}
&(\mathrm{i})\ \forall X \in \mathbb{G} \quad : \quad \mathbf{A}\left(\mathbf{X}^{\#}\right) = X \\
&(\mathrm{ii})\ \forall \mathfrak{g} \in G \quad : \quad \mathfrak{g}^{*}\mathbf{A} = \mathfrak{g}^{-1}\,\mathbf{A}\,\mathfrak{g}
\end{aligned}
$$

$$H_u P \equiv \{\mathbf{X} \in T_u P \mid \mathbf{A}(\mathbf{X}) = 0\}$$

$$\mathbf{A} = \mathfrak{g} \cdot \mathscr{A} \cdot \mathfrak{g}^{-1} + d\mathfrak{g} \cdot \mathfrak{g}^{-1}$$

Rheonomy of superspace



PRINCIPLE OF RHEONOMY

$$\frac{\partial}{\partial\theta}\text{bosonic field} = \frac{\partial}{\partial x}\text{fermionic field}$$
$$\frac{\partial}{\partial\theta}\text{fermionic field} = \frac{\partial}{\partial x}\text{bosonic field}$$

The principle of rheonomy is reminiscent of the Cauchy-Riemann equations satisfied by the real and imaginary parts of analytic functions.
Hence it encodes a sort of analyticity condition for the superconnections that constitute the field content of supergravity theories.

$$f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$$
$$\frac{\partial}{\partial y}u(x,y) = \frac{\partial}{\partial x}v(x,y)$$
$$\frac{\partial}{\partial x}v(x,y) = -\frac{\partial}{\partial x}u(x,y)$$

The Bianchi Identities

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities which play an even more fundamental role in constructing supergravity theories then they played in constructing General Relativity:

$$\mathscr{D}\mathfrak{R}^{ab} = 0$$

$$\mathscr{D}\mathfrak{T}^{a} + \mathfrak{R}^{ab} \wedge V_b - \mathrm{i}\overline{\psi} \wedge \Gamma^{a}\rho = 0$$

$$\mathscr{D}\rho + \frac{1}{4}\Gamma^{ab}\psi \wedge \mathfrak{R}^{ab} = 0$$

$$d\mathbf{F}^{[4]} - \overline{\psi} \wedge \Gamma_{ab}\rho \wedge V^{a} \wedge V^{b} + \overline{\psi}\Gamma_{ab}\psi \wedge \mathfrak{T}^{a} \wedge V^{b} = 0$$

$$\begin{aligned} & d\mathbf{F}^{[7]} - \mathrm{i}\overline{\psi} \wedge \Gamma_{a_1\cdots a_5}\rho \wedge V^{a_1} \wedge \cdots \wedge V^{a_5} \\ & \quad - \frac{5}{3}\mathrm{i}\overline{\psi} \wedge \Gamma_{a_1\cdots a_5}\psi \wedge \mathfrak{T}^{a_1} \wedge V^{a_2} \wedge \cdots \wedge V^{a_5} \\ & \quad - 15\overline{\psi} \wedge \Gamma_{ab}\rho \wedge V^{a} \wedge V^{b} \wedge \mathbf{F}^{[4]} - 15\mathbf{F}^{[4]} \wedge \mathbf{F}^{[4]} = 0 \end{aligned}$$

Luigi Bianchi
1856-1928

The rheonomic solution of Bianchi.s

$$\mathfrak{T}^a = 0$$

$$\mathbf{F}^{[4]} = F_{a_1 \ldots a_4} V^{a_1} \wedge \cdots \wedge V^{a_4}$$

$$\mathbf{F}^{[7]} = \frac{1}{84} F^{a_1 \ldots a_4} V^{b_1} \wedge \cdots \wedge V^{b_7} \varepsilon_{a_1 \ldots a_4 b_1 \ldots b_7}$$

$$\rho = \rho_{a_1 a_2} V^{a_1} \wedge V^{a_2} - \mathrm{i}\frac{1}{2}\left(\Gamma^{a_1 a_2 a_3} \psi \wedge V^{a_4} + \frac{1}{8}\Gamma^{a_1 \ldots a_4 m} \psi \wedge V^m\right) F^{a_1 \ldots a_4}$$

$$\mathfrak{R}^{ab} = R^{ab}{}_{cd} V^c \wedge V^d + \mathrm{i}\rho_{mn}\left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a}\delta^{b]c} + 2\Gamma^{ab[m}\delta^{n]c}\right)\psi \wedge V^c$$

$$+ \overline{\psi} \wedge \Gamma^{mn} \psi F^{mnab} + \frac{1}{24}\overline{\psi} \wedge \Gamma^{abc_1 \ldots c_4} \psi F^{c_1 \ldots c_4}$$

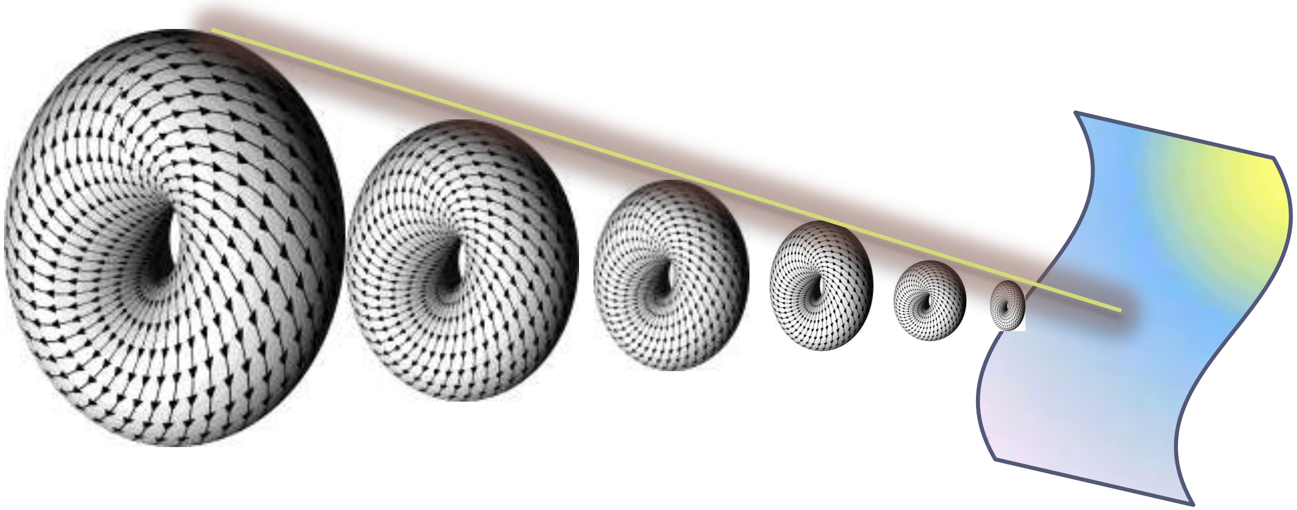
The field equations = integrability conditons

$$0 = \mathscr{D}_m F^{mc_1c_2c_3} + \frac{1}{96}\varepsilon^{c_1c_2c_3a_1a_8} F_{a_1\ldots a_4} F_{a_5\ldots a_8}$$

$$0 = \Gamma^{abc} \rho_{bc}$$

$$R^{am}{}_{cm} = 6 F^{ac_1c_2c_3} F^{bc_1c_2c_3} - \frac{1}{2}\delta^a_b F^{c_1\ldots c_4} F^{c_1\ldots c_4}$$

Since there are no auxiliary fields and the supersymmetry algebra closes only on-shell the rheonomic parameterization of the curvatures yields thee full result. Field equations, namely dynamics, follows directly from the Bianchi identities. I do not need the action which in any case exists.



Supergravity & Holography Mini Course @ Viña del Mar

Pietro Fré

LECTURE 3 : Chern Simons N=1
Gauge Theories in D=3

Rheonomic construction of matter coupled $\mathscr{N}=2$ gauge theories in $D=3$

A case with auxiliary fields

The supergeometry of $D=3$, $\mathscr{N}$=2 rigid superspace

$D=3$, $\mathscr{N}$–extended superspace is viewed as the following supercoset manifold:

$$\mathscr{M}_3^{\mathscr{N}} = \frac{\mathrm{ISO}(1,2|\mathscr{N})}{\mathrm{SO}(1,2)} \equiv \frac{Z\left[\mathrm{ISO}(1,2|\mathscr{N})\right]}{\mathrm{SO}(1,2)\times\mathbb{R}^{\mathscr{N}(\mathscr{N}-1)/2}}$$

where $\mathrm{ISO}(1,2|\mathscr{N})$ is the $\mathscr{N}$–extended Poincaré supergroup in three–dimensions. Its superalgebra is the Inonü-Wigner contraction of the superalgebra $\mathrm{Osp}(\mathscr{N}|4)$ spanned by the generators J_m, P_m, q^i. The central extension $Z\left[\mathrm{ISO}(1,2|\mathscr{N})\right]$, which is not contained in the contraction of $\mathrm{Osp}(\mathscr{N}|4)$, is obtained by adjoining to $\mathrm{ISO}(1,2|\mathscr{N})$ the central charges that generate the subalgebra $\mathbb{R}^{\mathscr{N}(\mathscr{N}-1)/2}$. Specializing our analysis to the case $\mathscr{N}$=2, we define the new generators:

$$\begin{cases} Q &=& \sqrt{2}q^- = (q^1 - iq^2) \\ Q^c &=& \sqrt{2}iq^+ = i(q^1 + iq^2) \\ Z &=& Z^{12} \end{cases}$$

The left invariant one–form Ω on $\mathscr{M}_3^{\mathscr{N}}$ is the following object:

$$\Omega = e^m P_m - \tfrac{1}{2}\omega^{mn} J_{mn} + \overline{\psi^c} Q - \overline{\psi} Q^c + \mathfrak{B} Z .$$

The superalgebra ISO$(1,2|\mathrm{N})$ defines all the structure constants apart from those relative to the central charge that are trivially determined. Hence we can write:

$$\begin{aligned} d\Omega - \Omega \wedge \Omega \;\; = \;\; & \left(de^m - \omega^m_{\ n} \wedge e^n + i\overline{\psi} \wedge \gamma^m \psi + i\overline{\psi}^c \wedge \gamma^m \psi^c\right) P_m \\ & -\tfrac{1}{2}\left(d\omega^{mn} - \omega^m_{\ p} \wedge \omega^{pn}\right) J_{mn} \\ & + \left(d\overline{\psi}^c + \tfrac{1}{2}\omega^{mn} \wedge \overline{\psi}^c \gamma_{mn}\right) Q \\ & + \left(d\overline{\psi} - \tfrac{1}{2}\omega^{mn} \wedge \overline{\psi}\gamma_{mn}\right) Q^c \\ & + \left(d\mathfrak{B} + i\overline{\psi}^c \wedge \psi^c - i\overline{\psi} \wedge \psi\right) Z \end{aligned}$$

Imposing the Maurer-Cartan equation $d\Omega - \Omega \wedge \Omega = 0$ is equivalent to imposing flatness in superspace, i.e. global supersymmetry. So we have

$$
\begin{aligned}
0 &= T^m \equiv \mathscr{D} e^m + \mathrm{i}\,(\overline{\psi}_c \wedge \gamma^m \psi_c + \overline{\psi} \wedge \gamma^m \psi) \\
0 &= R^{mn} \equiv d\omega^{mn} - \omega^{mp} \wedge \omega^{pn} \\
0 &= \rho \equiv d\psi + \frac{1}{2}\omega^{mn} \wedge \gamma_{mn} \psi \\
0 &= \rho_c \equiv d\psi_c - \frac{1}{2}\omega^{mn} \wedge \gamma_{mn} \psi_c \\
0 &= RZ \equiv d\mathfrak{B} + \mathrm{i}\,(\overline{\psi}_c \wedge \psi_c - \overline{\psi} \wedge \psi)
\end{aligned}
$$

In rigid supersymmetry we do not have to find a rheonomic parameterrization for the extended super Poincaré curvatures. They are all zero!

We have to find a rheonomic parameterization of the curvatures (derivatives) of matter fields.

The gauge multiplet

$$\text{vect. mult.} = \left\{ \underbrace{\mathscr{A}^\Lambda}_{\text{gauge one-form}}, \underbrace{\lambda^\Lambda, \lambda_c^\Lambda}_{\text{gauginos}}, \underbrace{M^\Lambda}_{\text{phys. scalar}} \underbrace{D^\Lambda}_{\text{aux. scalar}} \right\}$$

$$\begin{aligned} \mathfrak{F} &\equiv d\mathscr{A} + \mathscr{A} \wedge \mathscr{A} = \mathfrak{F}^\Lambda t_\Lambda \\ \mathfrak{F}^\Lambda &= \mathrm{d}\mathscr{A}^\Lambda + f^\Lambda{}_{\Delta\Sigma} \mathscr{A}^\Delta \wedge \mathscr{A}^\Sigma \end{aligned}$$

The off-shell rheonomic parametrization of the vector multiplet *curvatures*, consistent with the Bianchi identities is given below:

$$\begin{aligned} \mathfrak{F}^\Lambda &= F^\Lambda_{mn} e^M \wedge e^n - \mathrm{i}\overline{\psi}_c \gamma_m \lambda^\Lambda \wedge e^m - \mathrm{i}\overline{\psi} \gamma_m \lambda_c^\Lambda \wedge e^m \\ &\quad - \mathrm{i}M^\Lambda (\overline{\psi} \wedge \psi - \overline{\psi}_c \wedge \psi) \\ \nabla\lambda^\Lambda \equiv \mathrm{d}\lambda^\Lambda + [\mathscr{A}, \lambda]^\Lambda &= \nabla_m \lambda^\Lambda e^m + \nabla_m M^\Lambda \gamma^m \psi_c - F^\Lambda_{mn} \gamma^{mn} \psi_c + \mathrm{i}D^\Lambda \psi_c \\ \nabla\lambda_c^\Lambda \equiv \mathrm{d}\lambda_c^\Lambda + [\mathscr{A}, \lambda_c]^\Lambda &= \nabla_m \lambda_c^\Lambda e^m - \nabla_m M^\Lambda \gamma^m \psi - F^\Lambda_{mn} \gamma^{mn} \psi - \mathrm{i}D^\Lambda \psi \\ \nabla M^\Lambda \equiv \mathrm{d}M^\Lambda + [\mathscr{A}, M]^\Lambda &= \nabla_m M_c^\Lambda e^m + \mathrm{i}\overline{\psi} \lambda_c^\Lambda - \mathrm{i}\overline{\psi}_c \lambda^\Lambda \\ \nabla D^\Lambda \equiv \mathrm{d}D^\Lambda + [\mathscr{A}, D]^\Lambda &= \nabla_m D_c^\Lambda e^m + \overline{\psi} \gamma^m \nabla_m \lambda_c^\Lambda - \overline{\psi}_c \gamma^m \nabla_m \lambda^\Lambda \\ &\quad - \mathrm{i}\overline{\psi}\ [\lambda_c, M]^\Lambda - \mathrm{i}\overline{\psi}_c\ [\lambda, M]^\Lambda \end{aligned}$$

The chiral Wess Zumino multiplets

$$\text{WZ. mult.} = \left\{ \underbrace{z^i}_{\text{complex scalars}} \, , \, \underbrace{\chi^i, \chi^i_c}_{\text{chiralinos}} \, , \, \underbrace{\mathcal{H}^i}_{\text{complex aux fields}} \right\}$$

The complex scalar fields z^i parameterize a Kähler manifold $\mathcal{M}_K$ whose geometry is determined by a Kähler potential $\mathcal{K}(z, \bar{z})$ yielding as usual the metric:

$$g_{ij^\star} = \partial_i \partial_{j^\star} \mathcal{K}$$

The continuous isometries (if any) of this metric are generated by holomorphic Killing vectors $k^i_\Lambda(z)$ according to:

$$z^i \mapsto z^i + \varepsilon^\Lambda k^i_\Lambda(z)$$

and the vector multiplets can be used to gauge such symmetries and make them local.

Additional essential items in the construction of the theory are the moment maps defined by the following equation:

$$k^i_\Lambda = \mathrm{i}\, g^{ij^\star} \partial_{j^\star} \mathcal{P}_\Lambda \quad ; \quad k^i_\Lambda = -\mathrm{i}\, g^{ij^\star} \partial_{j^\star} \mathcal{P}_\Lambda$$

The gauging of isometries is obtained modifying the natural Levi Civita connection of the Kaehler manifold

$$\begin{aligned}
\nabla z^i &\equiv dz^i + \mathscr{A}^\Lambda k^i_\Lambda(z) \\
\nabla \chi^i &\equiv d\chi^i + \widehat{\Gamma}^i{}_j \chi^j \quad ; \quad \widehat{\Gamma}^i{}_j = \Gamma^i{}_j + \mathscr{A}^\Lambda \partial_j k^i_\Lambda \\
\nabla \chi^{j^\star} &\equiv d\chi^{j^\star} + \widehat{\Gamma}^{j^\star}{}_{k^\star} \chi^{k^\star} \quad ; \quad \widehat{\Gamma}^{j^\star}{}_{k^\star} = \Gamma^{j^\star}{}_{k^\star} + \mathscr{A}^\Lambda \partial_j k^i_\Lambda \\
\nabla \mathscr{H}^i &\equiv d\mathscr{H}^i + \widehat{\Gamma}^i{}_j \mathscr{H}^j
\end{aligned}$$

THE RHEONOMIC PARAMETERIZATION OF MATTER CURVATURES

$$\begin{aligned}
\nabla z^i &= \nabla_m z^i e^m + 2\overline{\psi}_c \chi^i \\
\nabla \overline{z}^{i^\star} &= \nabla_m \overline{z}^{i^\star} e^m + 2\overline{\psi} \chi^{i^\star}_c \\
\nabla \chi^i &= \nabla_m \chi^i e^m - \mathrm{i} \nabla_m z^i \gamma^m \psi_c + \mathscr{H}^i \psi - \mathrm{i} M^\Lambda k^i_\Lambda \psi_c \\
\nabla \chi^{j^\star} &= \nabla_m \chi^{j^\star} e^m + \mathrm{i} \nabla_m \overline{z}^{j^\star} \gamma^m \psi - \overline{\mathscr{H}}^{j^\star} \psi_c + \mathrm{i} M^\Lambda k^i_\Lambda \psi
\end{aligned}$$

The rheonomic lagrangian

$$\mathcal{L}^{\mathcal{N}=2}_{rheo} = \mathcal{L}^{gauge}_{rheo} + \mathcal{L}^{chiral}_{rheo}$$
$$\mathcal{L}^{gauge}_{rheo} = \mathcal{L}^{Maxwell}_{rheo} + \mathcal{L}^{Chern-Simons}_{rheo} + \mathcal{L}^{Fayet-Iliopoulos}_{rheo}$$
$$\mathcal{L}^{chiral}_{rheo} = \mathcal{L}^{Kahler}_{rheo} + \mathcal{L}^{superpotential}_{rheo}$$

It is a long matter to write the complete rheonomic lagrangian and even a longer task to determine it. Yet it is just a boring but straightforward algorithm.

$$\begin{aligned}
\mathcal{L}^{Maxwell}_{rheo} = \; & \mathbf{e}\,\mathrm{Tr}\Big\{-F^{mn}\left[\mathfrak{F} + i\overline{\psi}^c\gamma_m\lambda \wedge e^m + i\overline{\psi}\gamma_m\lambda^c \wedge e^m - 2iM\overline{\psi}\wedge\psi\right]\wedge e^p\varepsilon_{mnp} \\
& +\tfrac{1}{6}F_{qr}F^{qr}e^m\wedge e^n\wedge e^p\varepsilon_{mnp} - \tfrac{1}{4}i\varepsilon_{mnp}\left[\nabla\overline{\lambda}\gamma^m\lambda + \nabla\overline{\lambda}^c\gamma^m\lambda^c\right]\wedge e^n\wedge e^p \\
& \tfrac{1}{2}\varepsilon_{mnp}\Phi^m\left[\nabla M - i\overline{\psi}\lambda^c + i\overline{\psi}^c\lambda\right]\wedge e^n\wedge e^p - \tfrac{1}{12}\Phi^d\Phi_d\,\varepsilon_{mnp}e^m\wedge e^n\wedge e^p \\
& +\nabla M\wedge\overline{\psi}^c\gamma_c\lambda\wedge e^p - \nabla M\wedge\overline{\psi}\gamma_p\lambda^c\wedge e^p \\
& +\mathfrak{F}\wedge\overline{\psi}^c\lambda + \mathfrak{F}\wedge\overline{\psi}\lambda^c + \tfrac{1}{2}i\overline{\lambda}^c\lambda\overline{\psi}^c\wedge\gamma_m\psi\wedge e^m + \tfrac{1}{2}i\overline{\lambda}\lambda^c\overline{\psi}\wedge\gamma_m\psi^c\wedge e^m \\
& +\tfrac{1}{12}D^2e^m\wedge e^n\wedge e^p\varepsilon_{mnp} - 2i(\overline{\psi}\wedge\psi)M\wedge\left[\overline{\psi}^c\lambda + \overline{\psi}\lambda^c\right] \\
& -\tfrac{1}{6}M\left[\overline{\lambda},\lambda\right]e^m\wedge e^n\wedge e^p\,\varepsilon_{mnp}\Big\}
\end{aligned}$$

The space–time Lagrangian of the Maxwell-Chern-Simons theory and some of its applications

The space–time lagrangian,

$$\mathcal{L}_{st}^{\mathcal{N}=2} = \mathcal{L}_{kinetic} + \mathcal{L}_{2fermi} + \mathcal{L}_{potential}$$

$$\begin{aligned}
\mathcal{L}_{kin} \;=\; & -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A} + \frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right) - \mathbf{e}\,\mathrm{Tr}\,(F^{pq}\,\mathfrak{F})\wedge e^r\,\varepsilon_{pqr} \\
& + \left(\frac{1}{2}g_{ij^\star}\left(\Pi^{m|i}\,\nabla\overline{z}^{j^\star} + \overline{\Pi}^{m|j^\star}\,\nabla z^i\right) + \mathbf{e}\,\mathrm{Tr}\,(\Phi^m\,\mathrm{d}M)\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp} \\
& + \left(\mathbf{e}\frac{1}{6}F_{qr}F^{qr} - \frac{1}{6}g_{ij^\star}\,\Pi^{m|i}\,\overline{\Pi}^{m|j^\star} - \mathbf{e}\frac{1}{12}\Phi^p\,\Phi_p\right)e^m\wedge e^n\wedge e^p\,\varepsilon_{mnp} \\
& \left(\mathbf{e}\,\mathrm{Tr}\left(\nabla\overline{\lambda}\,\gamma^n\lambda + \nabla\overline{\lambda}_c\,\gamma^n\lambda_c\right) + \mathrm{i}\frac{1}{2}g_{ij^\star}\left(\overline{\chi}^{j^\star}\,\gamma^n\,\nabla\chi i + \overline{\chi}^i_c\,\gamma^n\,\nabla\chi i^\star{}_c\right)\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}
\end{aligned}$$

Some fields are immediately eliminated through their own algebraic equation

$$\Pi^{m|i} = \nabla_m z^i \quad ; \quad \overline{\Pi}^{m|j^\star} = \nabla_m \overline{z}^{j^\star} \qquad \Phi^\Lambda_m = \nabla_m M^\Lambda$$

$$\mathfrak{F} = F_{pq}\,e^p\wedge e^q$$

The other terms of the lagrangian

$$\begin{aligned}
\mathscr{L}_{2fermi} &= \left(-\frac{1}{3} M^{\Lambda} \left(\partial_i k^j_{\Lambda} g_{j\ell^\star} \overline{\chi}^{\ell^\star} \chi^i + \partial_{i^\star} k^{j^\star}_{\Lambda} g_{j\ell^\star} \overline{\chi}^{\ell}_c \chi^{i^\star}_c \right) + \frac{\alpha}{3} \left(\overline{\lambda}^{\Lambda} \lambda^{\Sigma} + \overline{\lambda}^{\Lambda}_c \lambda^{\Sigma}_c \right) \kappa_{\Lambda\Sigma} \right. \\
&\quad + \mathrm{i} \frac{1}{3} \left(\overline{\chi}^{j^\star}_c \lambda^{\Lambda} k^i_{\Lambda} - \overline{\chi}^i_c \lambda^{\Lambda} k^{j^\star}_{\Lambda} \right) g_{ij^\star} - \mathbf{e} \frac{1}{6} \mathrm{Tr} \left(M \left([\lambda\,,\lambda] + [\lambda_c\,,\lambda_c] \right) \right) \\
&\quad \left. + \frac{1}{6} \left(\partial_i \partial_j W \overline{\chi}^i_c \chi^j + \partial_{i^\star} \partial_{j^\star} \overline{W} \overline{\chi}^{i^\star} \chi^{j^\star}_c \right) \right) \epsilon_{mnp} e^m \wedge e^n \wedge e^p
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{pot} &= -V\left(M, P, \mathscr{H}, z, \overline{z}\right) \epsilon_{mnp} e^m \wedge e^n \wedge e^p \\
V\left(M, P, \mathscr{H}, z, \overline{z}\right) &= \left(\frac{\alpha}{3} M^{\Lambda} \kappa_{\Lambda\Sigma} - \frac{1}{6} \mathscr{P}_{\Sigma}(z, \overline{z}) + \frac{1}{6} \mathfrak{f}_I \mathfrak{C}^I_{\Sigma} \right) D^{\Sigma} + \frac{1}{6} M^{\Lambda} M^{\Sigma} k^i_{\Lambda} k^{j^\star}_{\Sigma} g_{ij^\star} \\
&\quad + \frac{1}{6} \left(\mathscr{H}^i \partial_i W + \mathscr{H}^{\ell^\star} \partial_{\ell^\star} \overline{W} \right) - \frac{1}{6} g_{i\ell^\star} \mathscr{H}^i \mathscr{H}^{\ell^\star} - \frac{1}{2} \mathbf{e} \, \kappa_{\Lambda\Sigma} D^{\Lambda} D^{\Sigma}
\end{aligned}$$

In the above equation the vectors $\mathfrak{C}^I_{\Sigma}$ $(I = 1, \ldots, r)$ project onto the r independent generators of the center of the gauge Lie algebra $Z(\mathbb{G})$. For each of these generators one can add a separately supersymmetric invariant term, named Fayet Iliopoulos term [53], which is just linear in the corresponding auxiliary fields $D_I \equiv \mathfrak{C}^I_{\Sigma} D^{\Sigma}$. Namely we have:

Note that e, α and f^I are the coefficients of separate off-shell supersymmetric invariants

$\mathcal{N} = 2$ Pure Chern Simons Gauge Theory

When $\mathbf{e} = 0$, the lagranian takes the following form:

$$\begin{aligned}
\mathcal{L}_{CSoff} \;=\; & -\alpha \,\mathrm{Tr}\left(\mathfrak{F} \wedge \mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) + \left(\frac{1}{2} g_{ij^\star} \Pi^{m|i} \nabla \bar{z}^{j^\star} + \overline{\Pi}^{m|j^\star} \nabla z^i\right) \wedge e^n \wedge e^p \,\varepsilon_{mnp} \\
& -\frac{1}{6} g_{ij^\star} \Pi^{m|i} \overline{\Pi}^{m|j^\star} e^r \wedge e^s \wedge e^t \,\varepsilon_{rst} \\
& +\mathrm{i}\frac{1}{2} g_{ij^\star} \left(\overline{\chi}^{j^\star} \gamma^n \nabla \chi i + \overline{\chi}^i_c \gamma^n \nabla \chi i^\star{}_c\right) \wedge e^n \wedge e^p \,\varepsilon_{mnp} \\
& \left(-\frac{1}{3} M^\Lambda \left(\partial_i k^j_\Lambda g_{j\ell^\star} \overline{\chi}^{\ell^\star} \chi^i + \partial_{i^\star} k^{j^\star}_\Lambda g_{j\ell^\star} \overline{\chi}^\ell_c \chi^{i^\star}_c\right) + \frac{\alpha}{3}\left(\overline{\lambda}^\Lambda \lambda^\Sigma + \overline{\lambda}^\Lambda_c \lambda^\Sigma_c\right) \kappa_{\Lambda\Sigma}\right. \\
& +\mathrm{i}\frac{1}{3}\left(\overline{\chi}^{j^\star}_c \lambda^\Lambda k^i_\Lambda - \overline{\chi}^i_c \lambda^\Lambda k^{j^\star}_\Lambda\right) g_{ij^\star} \\
& \left.+\frac{1}{6}\left(\partial_i \partial_j W \overline{\chi}^i_c \chi^j + \partial_{i^\star} \partial_{j^\star} \overline{W} \,\overline{\chi}^{i^\star} \chi^{j^\star}_c\right)\right) \wedge e^n \wedge e^p \,\varepsilon_{mnp} \\
& -V\left(M, D, \mathcal{H}, z, \bar{z}\right) \varepsilon_{mnp} e^m \wedge e^n \wedge e^p
\end{aligned}$$

Some physical fields become lagrangian multipliers and can be eliminated by their eq. s of motion.

$$\begin{aligned}
V\left(M, D, \mathcal{H}, z, \bar{z}\right) \;=\; & \left(\frac{\alpha}{3} M^\Lambda \kappa_{\Lambda\Sigma} - \frac{1}{6}\mathcal{P}_\Sigma(z,\bar{z}) + \frac{1}{6}\mathfrak{f}_I \mathfrak{C}^I_\Sigma\right) D^\Sigma + \frac{1}{6} M^\Lambda M^\Sigma k^i_\Lambda k^{j^\star}_\Sigma g_{ij^\star} \\
& + \frac{1}{6}\left(\mathcal{H}^i \partial_i W + \mathcal{H}^{\ell^\star} \partial_{\ell^\star} \overline{W}\right) - \frac{1}{6} g_{i\ell^\star} \mathcal{H}^i \mathcal{H}^{\ell^\star}
\end{aligned}$$

Let us consider what happens in the sector of the scalars

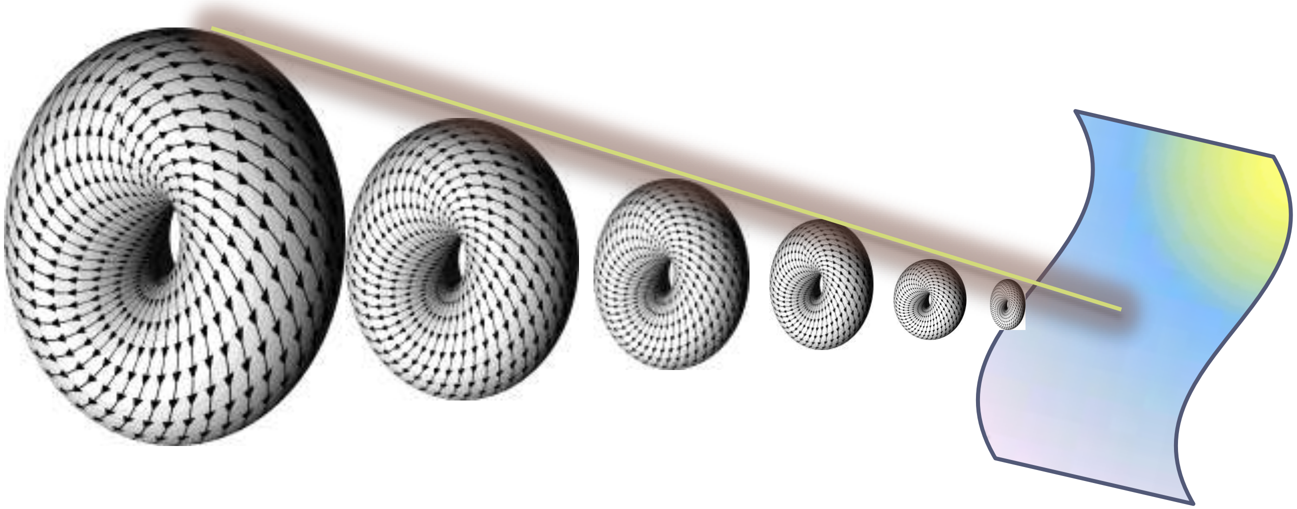
$$M^{\Lambda} = \frac{1}{2\alpha} \kappa^{\Lambda\Sigma} \left(\mathscr{P}_{\Sigma} - \mathfrak{f}_I \mathfrak{C}^I_{\Sigma} \right)$$

$$\mathscr{H}^i = g^{ij^\star} \partial_{j^\star} \overline{W} \quad ; \quad \overline{\mathscr{H}}^{j^\star} = g^{ij^\star} \partial_i W$$

$$D^{\Lambda} = -\frac{1}{\alpha} \kappa^{\Lambda\Gamma} g_{ij^\star} k^i_{\Gamma} k^{j^\star}_{\Sigma} M^{\Sigma} = -\frac{1}{2\,\alpha^2} g_{ij^\star} \kappa^{\Lambda\Gamma} k^i_{\Gamma} k^{j^\star}_{\Sigma} \kappa^{\Sigma\Delta} \left(\mathscr{P}_{\Delta} - \mathfrak{f}_I \mathfrak{C}^I_{\Delta} \right)$$

$$V(z,\bar{z}) = \frac{1}{6} \left(\partial_i W \, \partial_{j^\star} \overline{W} g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma} \left(\mathscr{P}_{\Lambda} - \mathfrak{f}_I \mathfrak{C}^I_{\Lambda} \right) \left(\mathscr{P}_{\Sigma} - \mathfrak{f}_I \mathfrak{C}^I_{\Sigma} \right) \right)$$

$$\mathbf{m}^{\Lambda\Sigma}(z,\bar{z}) \equiv \frac{1}{4\alpha^2} \kappa^{\Lambda\Gamma} \kappa^{\Sigma\Delta} k^i_{\Gamma} k^{j^\star}_{\Delta} g_{ij^\star}$$



Supergravity & Holography
Mini Course @ Viña del Mar

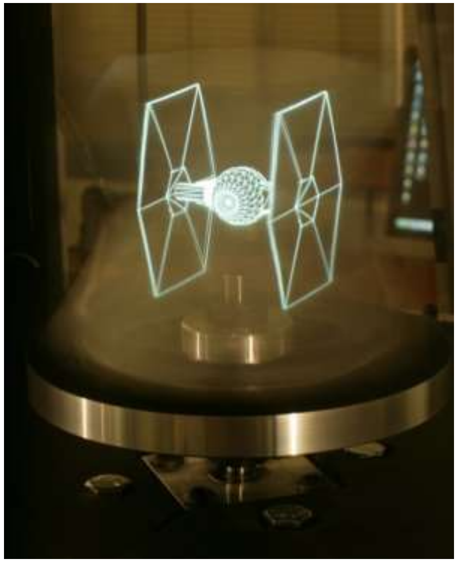
Pietro Fré

LECTURE 4 : Sasakian manifolds

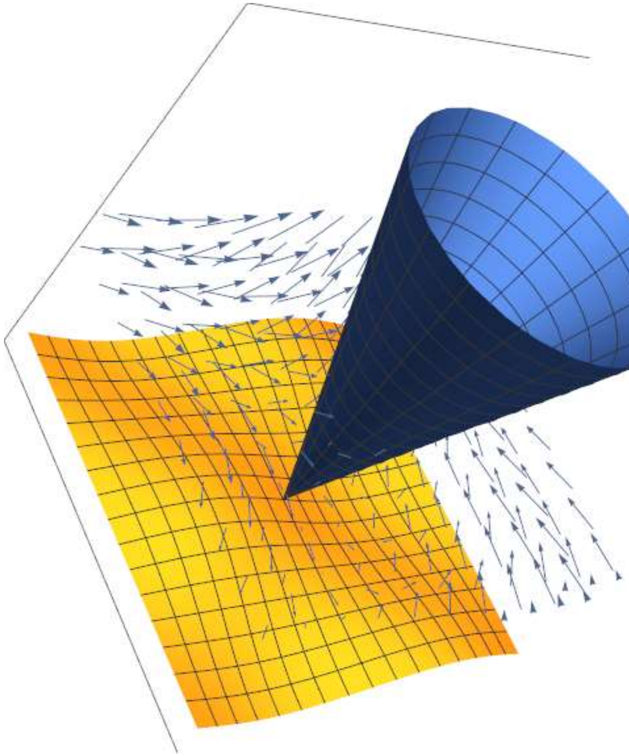
Let us now go back to geometry

Sasakian Manifolds & Orbifolds

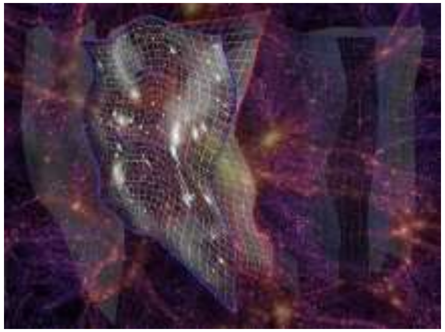
We consider now holography for M2 branes.



We will consider solutions of D=11 Sugra that can be interpreted as M2-branes where we have a d=3 world sheet and a complementary 8 dimensional space that is the metric cone over a Sasakian compact space in 7-dimensions.



The notion of Sasakian manifolds we explain in the next slide.
Their geometrical characterization is what ensures that on the brane world-sheet we have an N=2 gauge theory



What sasakian means is visually summarized in the following table.

base of the fibration	projection	7-manifold	metric cone
$\mathscr{B}_6$	$\xleftarrow{\pi}$	$\mathscr{M}_7$	$\mathscr{C}(\mathscr{M}_7)$
$\Updownarrow$	$\forall p \in \mathscr{B}_6 \quad \pi^{-1}(p) \sim \mathbb{S}^1$	$\Updownarrow$	$\Updownarrow$
Kähler K_3		sasakian	Kähler Ricci flat K_4

$$\pi \quad : \quad \mathscr{M}_7 \xrightarrow{\mathbb{S}^1} K_3$$

$$ds^2_{\mathscr{M}_7} = (d\phi - \mathscr{A})^2 + g_{ij^\star} dz^i \otimes d\bar{z}^{j^\star}$$

$$ds^2_{\mathscr{C}(\mathscr{M}_7)} = dr^2 + 4e^2 r^2 ds^2_{\mathscr{M}_7}$$

Altogether the Ricci flat Kahler manifold K_4, which plays the role of transverse space to the M2-branes, is a line-bundle over the base manifold K_3:

$$\begin{array}{ccc} \pi & : & K_4 \longrightarrow K_3 \\ \forall p \in K_3 & & \pi^{-1}(p) \sim \mathbb{C}^\star \end{array}$$

All the manifolds listed in table 1 are sasakian in the sense described above. The $\mathfrak{so}(8)$-holonomy mentioned in this table is the holonomy of the Levi-Civita connection of the metric cone $\mathscr{C}(\mathscr{M}_7)$ which can be easily calculated from that of the $\mathscr{M}_7$-manifold relying on the following one-line construction. Define the vielbein of $\mathscr{C}(\mathscr{M}_7)$ in terms of the vielbein of $\mathscr{M}_7$ in the following way:

$$V^I = \left\{ \begin{array}{rcl} V^0 & = & dr \\ V^\alpha & = & e\, r\, \mathscr{B}^\alpha \end{array} \right. \qquad r \in \mathbb{R}_+ \tag{1.5}$$

where $ds^2_{\mathscr{M}_7} = \sum_{\alpha=1}^{7} \mathscr{B}^\alpha \otimes \mathscr{B}^\alpha$. The torsion equation:

$$dV^I + \Omega^{IJ} \wedge V^J = 0 \tag{1.6}$$

where Ω^{IJ} is the spin–connection of the metric cone, is solved by:

$$\begin{array}{rcl} \Omega^{\alpha\beta} & = & \mathscr{B}^{\alpha\beta} \\ \Omega^{0\beta} & = & -2\, e\, r\, \mathscr{B}^\beta \end{array}$$

having denoted by $\mathscr{B}^{\alpha\beta}$ the spin–connection of $\mathscr{M}_7$, namely $d\mathscr{B}^\alpha + \mathscr{B}^{\alpha\beta} \wedge \mathscr{B}^\beta = 0$.

Relation with AdS compactification of D=11 Supergravity

According to the summary of Kaluza–Klein supergravity presented in [37], Ω^{IJ} is the $\mathfrak{so}(8)$-connection whose holonomy decides the number of Killing spinor admitted by the $AdS_4 \times \mathscr{M}_7$ compactification of M-theory. When this holonomy vanishes we have the maximal number of preserved supersymmetries. When it is $SU(3) \subset SO(8)$ we have $\mathscr{N} = 2$. When it is $SU(2) \subset SO(8)$ we might in principle expect $\mathscr{N} = 4$, but we actually have only $\mathscr{N} = 3$, as firstly remarked by Castellani, Romans and Warner in 1985.

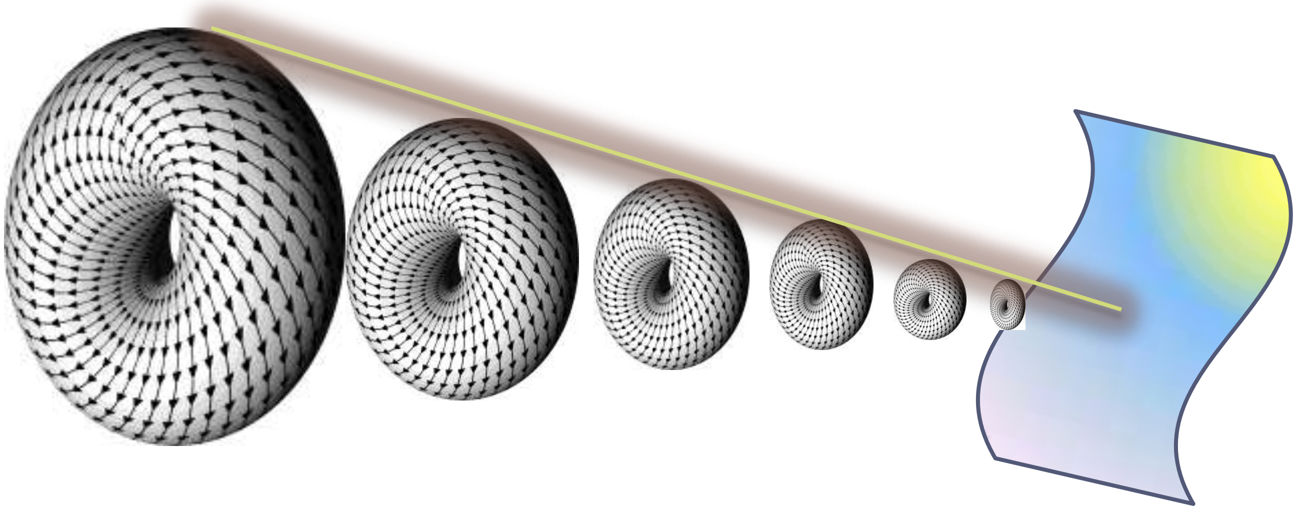
Sasakian homogenous 7-manifolds

$\mathcal{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle	Fibration
8	$\mathbb{S}^7$	$\frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}$	1	$\begin{cases} \mathbb{S}^7 \stackrel{\pi}{\Longrightarrow} \mathbb{P}^3 \\ \forall p \in \mathbb{P}^3 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$M^{1,1,1}$	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	$\mathrm{SU}(3)$	$\begin{cases} M^{1,1,1} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^2 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$Q^{1,1,1}$	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	$\mathrm{SU}(3)$	$\begin{cases} Q^{1,1,1} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$V^{5,2}$	$\frac{\mathrm{SO}(5)}{\mathrm{SO}(2)}$	$\mathrm{SU}(3)$	$\begin{cases} V^{5,2} \stackrel{\pi}{\Longrightarrow} M_a \sim \text{quadric in } \mathbb{P}^4 \\ \forall p \in M_a ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
3	$N^{0,1,0}$	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	$\mathrm{SU}(2)$	$\begin{cases} N^{0,1,0} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \\ \forall p \in \mathbb{P}^2 ; \pi^{-1}(p) \sim \mathbb{S}^3 \\ \hline N^{0,1,0} \stackrel{\pi}{\Longrightarrow} \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)} \\ \forall p \in \frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)} ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$

The role of algebraic geometry

In [11], it was emphasized that the fundamental geometrical clue to the field content of the *superconformal gauge theory* on the boundary is provided by the construction of the Kähler manifold K_4 as a holomorphic algebraic variety in some higher dimensional affine or projective space $\mathbb{V}_q$, plus a Kähler quotient. The equations identifying the algebraic locus in $\mathbb{V}_q$ are related with the superpotential W appearing in the $d = 3$ lagrangian, while the Kähler quotient is related with the D-terms appearing in the same lagrangian. The coordinates u, v of the space $\mathbb{V}_q$ are the scalar fields of the *superconformal gauge theory*, whose vacua, namely the set of extrema of its scalar potential, should be in one-to-one correspondence with the points of K_4. Going from one to multiple M2–branes just means that the coordinate u, v of $\mathbb{V}_q$ acquire color indices under a proper set of color gauge groups and are turned into matrices. In this way we obtain *quivers*.

[11] D. Fabbri, P. Fre', L. Gualtieri, C. Reina, A. Tomasiello, A. Zaffaroni and A. Zampa, *3-D superconformal theories from Sasakian seven manifolds: New nontrivial evidences for* AdS_4/CFT_3 Nucl. Phys. B **577** (2000) 547 [hep-th/9907219].



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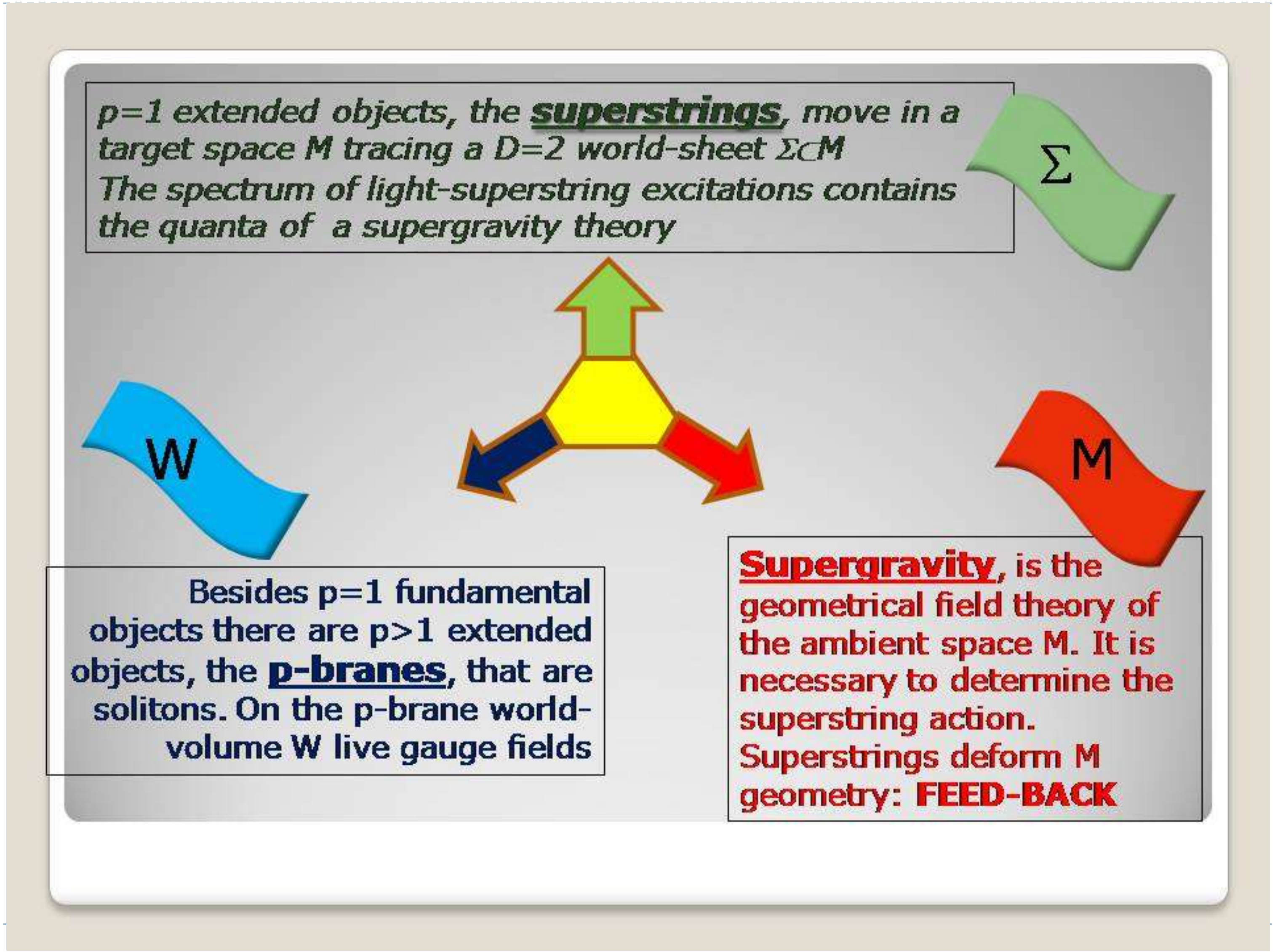
LECTURE 3 : M2-brane solutions

A triple of complementary viewpoints



Tu se' certo il cantor del trino regno,
Tu lo spirto magnanimo e sovrano
Cui, quasi cervo a puro fonte, io vegno.
Giovanni Marchetti

The triple Kingdom

$$\mathscr{A}_{part} = \underbrace{\int \sqrt{g_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}}\, d\tau}_{\int ds} + q \underbrace{\int A_{\mu}(x)\dot{x}^{\mu}\, d\tau}_{\int \mathbf{A}}$$

$$\mathscr{A}_{Max} = -\frac{1}{4}\int F^{\mu\nu} F_{\mu\nu}\, d^4x$$

$$J^{\mu}(x) = q \int \delta^{(4)}\big(x - x(\tau)\big)\, d\tau$$

Similarly we can vary the action $\mathscr{A}_{part}$ with respect to the metric $\delta g_{\mu\nu}$ and this yields a stress-energy tensor, also localized on the particle world-line, that provides a source for the gravitational field in Einstein equation.

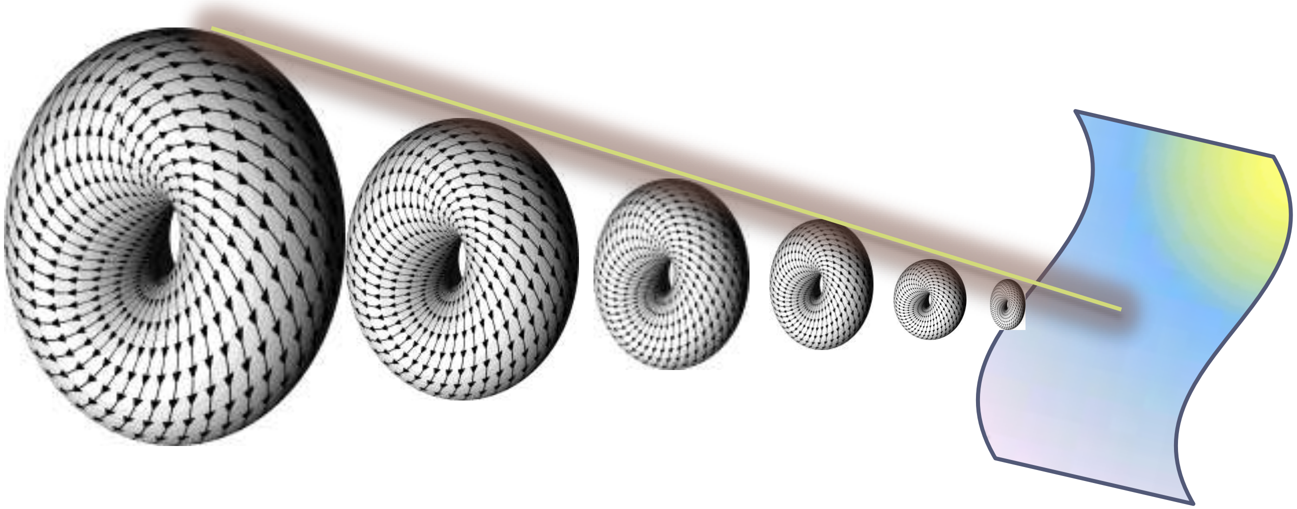
(A) If a field theory contains a gauge field that is a d-form $\mathbf{A}^{[d]}$ then, setting $p = d-1$, we can introduce a p-dimensional object which, by evolving through the ambient D-dimensional space-time $\mathscr{M}_D$, traces in this latter a d-dimensional world-volume (see Fig. 7.1):

$$\mathscr{W}_d \subset \mathscr{M}_D$$

The dynamics of such an extended object, which we name a p-brane, is described by an action given by a d-dimensional integral localized on the world-volume $\mathscr{W}_d$. Such a p-brane action is typically made of two terms

$$\mathscr{A}_{brane} = \mathscr{A}_{\text{Area}} + q \int_{\mathscr{W}_d} \mathbf{A}^{[d]}$$

the first term being the area of the world-volume or generalization thereof, the second, often named the Wess-Zumino term, being the integral of the form $\mathbf{A}^{[d]}$ on the world-volume.



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LECTURE 6 : The McKay correspondence & the Kronheimer construction

Based on the following work

- Recent work:
  - U. Bruzzo, A. Fino, P. Fré ArXiV:1710.01046 [hepth]
  - P. Fré, P.A. Grassi arXiv:1705.00752 [hepth]
  - P. Fré, arXiv:1601.02253
- Previous work
  - D. Fabbri, P.Fré, L. Gualtieri, C. Reina, A. Tommasiello, A. Zaffaroni, A. Zampa, hepth/9907219
  - D. Fabbri, P. Fré, L. Gualtieri, P. Termonia hepth/ 9905134
  - M. Billò, D. Fabbri, P, Fré, P. Merlatti, A. Zaffaroni, hepth/0005219
  - M. Bertolini, V.L. Campos, G. Ferretti, P. Fré, P. Salomonson, M. Trigiante, arXiv:hep-th/0106186
  - D. Anselmi, M. Billò, P. Fré, L. Girardello, A. Zaffaroni
    - hep-th/9304135

Generalized Kronheimer construction

For $\mathbb{C}^3/\Gamma$ there is a **generalized Kronheimer construction** of the resolution which is just taylored to define the building blocks of a gauge theory in D=3 or in D=4. This is based on a **generalized McKay correspondence.**

In D=3 it will be a Chern-Simons super gauge theory (N=2) corresponding to a M2 brane. In D=4 it will be an N=1 gauge theory associated with a D3-brane.

Before we inspect the Kronheimer construction in the perspective of Physics let us summarize some deep mathemtical results on the cohomolgy of the crepant resolutions of quotient singularities.

What we learn on cohomology from our friends mathematicians

Since years 1990s to the present time there has been a quite extended activity in the mathematical community of algebraic geometers on the issue of *crepant resolutions* $Y \to \mathbf{C}^n/\Gamma$ *(n=3 in particular)* and on *the McKay correspondence*. Some theorems have been established.

Important contributions have been given by: *Y.Ito, M. Reid, A. Craw, S.S. Roan, D. Markushevich, I. Dolgachev, A. Degeratu, T.Walpuski and others*.

The main and for physicists most challenging theorem is due to Ito & Reid and it is based on the notion of **age grading** which we briefly recall in the next slide.

The age grading

Let $\Gamma \subset$ SU(n) be a finite subgroup. Hence each of its group elements has a linear action on **C**^{n}: the $\mathcal{Q}$-representation.

$$\forall \gamma \in \Gamma : \quad \gamma.\vec{z} = \underbrace{\begin{pmatrix} \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}}_{\mathcal{Q}(\gamma)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

In a finite group every element has a finite order **p** : **γ^p =Id** (**p**=integer). Hence $\mathcal{Q}(\gamma)$ can be diagonalized and its eigenvalues are r-th roots of the unity. They will be as follows:

$$(\lambda_1, \ldots, \lambda_n) = \exp\left[\frac{2\pi\, i}{p} a_i\right] \quad ; \quad p > a_i \in \mathbb{N} \quad i = 1, \ldots, n$$

This introduces **age -vectors** $\mathbf{v} = \frac{1}{p}\{a_1, a_2, \ldots, a_n\}$

that are clearly properties of the entire **conjugacy class C of γ**

$$\text{age}(\gamma) = \frac{1}{p}\sum_{i=1}^{n} a_i$$ **AGE GRADING**

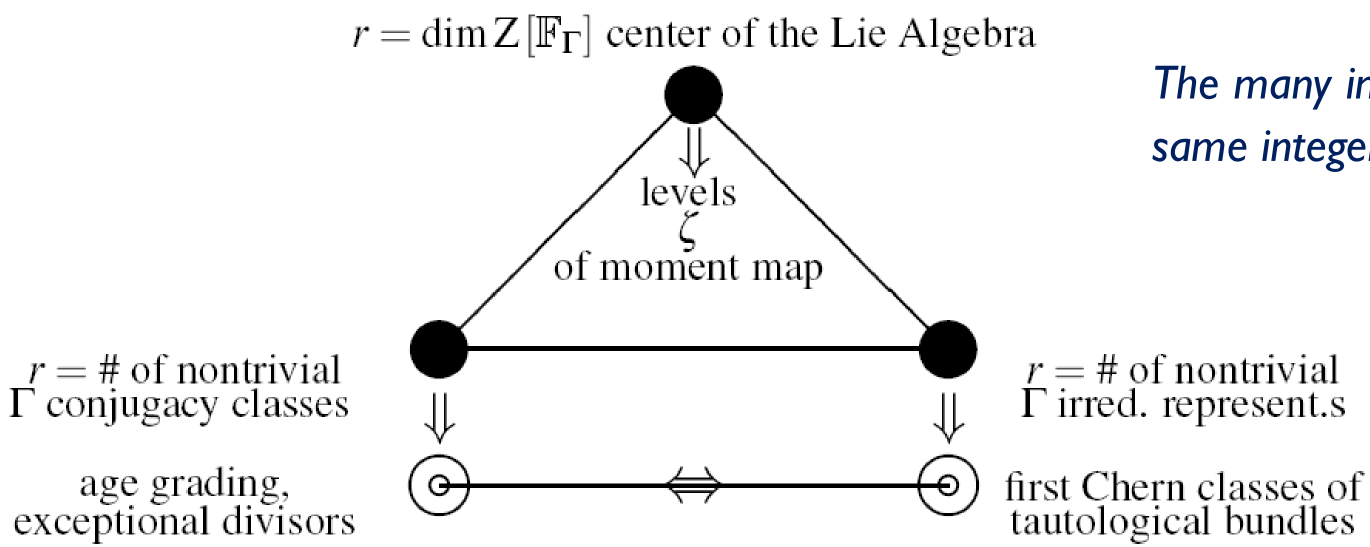
Ito Reid theorem

Theorem 4.1 *Let $Y \to \mathbb{C}^3/\Gamma$ be a crepant*[1] *resolution of a Gorenstein*[2] *singularity. Then we have the following relation between the de-Rham cohomology groups of the resolved smooth variety Y and the ages of Γ conjugacy classes:*

$$\dim H^{2k}(Y) = \#\ \textit{of age } k \textit{ conjugacy classes of } \Gamma$$

Furthermore $\dim H^{2k+1}(Y) = 0$ and the representatives of H^{2k} are actually (k,k)-forms

The age grading is not an intrinsic property of Γ, rather of its action on C^3



The many incarnations of the same integer number **r**

Terminology and some conclusions

There is a single class of age 0, namely the identity. The classes of age 1 are named junior classes. The classes of age 2 are named senior classes.

Junior classes are in one-to-one correspondence with a basis of generators of $\mathbf{H}^{(1,1)}$. These generators $\Omega^{(1,1)}_i$ can be regarded as the first Chern classes of as many line-bundles $\mathcal{L}_i$ and these line bundles correspond to as many divisors $\mathcal{D}_i$. These are the components of the exceptional divisor $\mathcal{D}_E$ created by the blow-up. When an $\Omega^{(1,1)}_i$ has compact support, by Poincaré duality it is dual to an $\Omega^{(2,2)}_i$ belonging to $\mathbf{H}^{(2,2)}$. These are in correspondence with the **senior classes**. **In other words the senior classes are in one-to-one correspondence with the compact components of the exceptional divisor**.

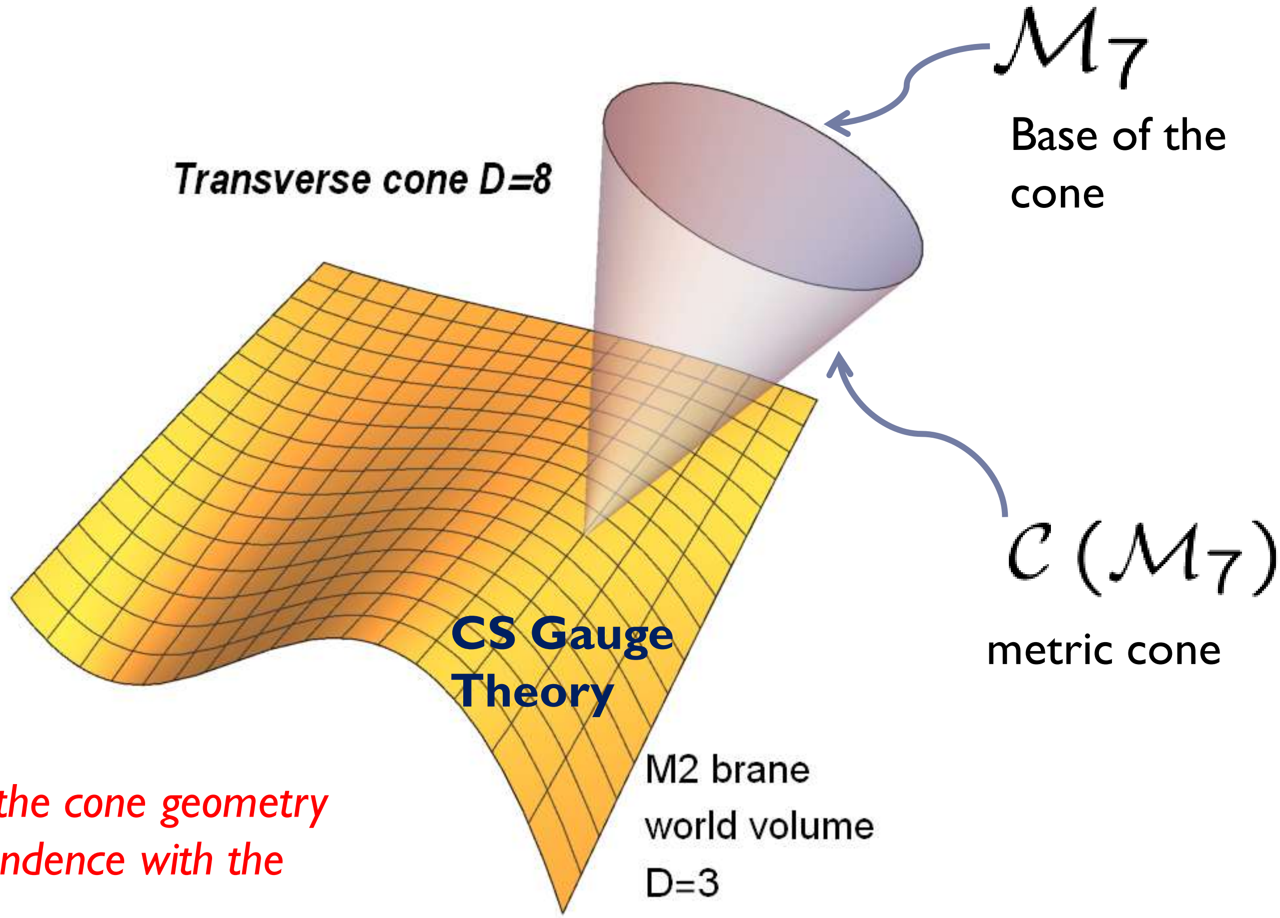
We have the 2-forms $\omega^{(1,1)}_i$ defined by **the generalized Kronheimer construction** and in one-to-one correspondence with the irreps of Γ. What is their precise relation with the $\Omega^{(1,1)}_i$ and the divisors $\mathcal{D}_i$ that are in one-to-one correspondence with the conjugacy classes? ***This pairing between irreps and conjugacy classes is of the outmost interest in Physics and we are working on its clarification.***

The scenario in Physics

Let us review the Physics section of our stage: Gauge/Gravity correspondence and branes

The AdS_4/CFT_3 scenario and some history: 1°

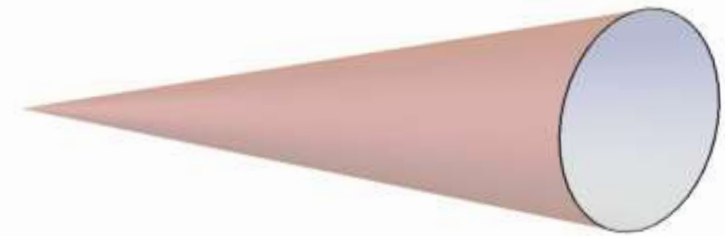
The fundamental issue is as follows.
Let us consider M2-brane solutions of D=11 SUGRA.
We have a Chern-Simons gauge theory on the D=3 world volume.
What can we learn on this CS theory, from the geometry of the transverse cone?



Essentially everything is fixed by the cone geometry and there is a beautiful correspondence with the mathem. theory of singularity resolutions.

The AdS$_4$/CFT$_3$ scenario and some history: 2°

The classical case studied 18 years ago in Fabbri *et al* hepth/9907219 corresponds to the case where the transverse cone is the metric cone on **a Sasakian homogeneous manifold G/H**



base of the fibration	projection	7-manifold	inclusion	metric cone
$\mathscr{B}_6$	$\overset{\pi}{\longleftarrow}$	$\mathscr{M}_7$	$\hookrightarrow$	$\mathscr{C}(\mathscr{M}_7)$
$\Updownarrow$	$\forall p \in \mathscr{B}_6 \quad \pi^{-1}(p) \sim \mathbb{S}^1$	$\Updownarrow$		$\Updownarrow$
Kähler K_3		Sasakian		Kähler Ricci flat K_4

At the beginning of the 80.s the Kaluza Klein spectra on

$$\mathrm{AdS}_4 \times \left(\frac{\mathrm{G}}{\mathrm{H}}\right)_7$$

had been extensively studied. After the advent of the AdS$_5$ /CFT$_4$ correspondence it was natural to study the **AdS$_4$ /CFT$_3$** correspondence utilizing the ample lore accumulated 15 years before. The Torino & Sissa group worked on that in the years 1998-2000.

M2-brane solution of D=11 SUGRA

The metric in D=11

$$ds^2_{11} = H(y)^{-\frac{2}{3}} \left(d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}\right) - H(y)^{\frac{1}{3}} \left(ds^2_{\mathcal{M}_8}\right)$$

where $$ds^2_{\mathcal{M}_8} = dy^I \otimes dy^J \, g_{IJ}(y)$$

The 3-form

$$\mathbf{A}^{[3]} \propto H(y)^{-1} \left(d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho \, \epsilon_{\mu\nu\rho}\right)$$

The harmonic function in d=8

$$\Box_{\mathcal{M}_8} H(y) = 0$$

and define

$$H(y) \equiv 1 - \frac{1}{r(y)^6}$$

$$\begin{cases} r \to \infty = \text{asymptotic flat limit} \\ r \to 0 = \text{near horizon limit} \end{cases}$$

More precisely

Let us consider the harmonic function as a map

$$\mathfrak{H} \quad : \quad \mathcal{M}_8 \to \mathbb{R}_+$$

This introduces a foliation into a one-parameter family of 7-manifolds

$$\forall r \in \mathbb{R}_+ \quad : \quad \mathcal{M}_7(r) \equiv \mathfrak{H}^{-1}(1 - r^{-6}) \subset \mathcal{M}_8$$

In order to have the possibility of residual supersymmetries we are interested in cases where M_8 is actually a Ricci-flat Kaehler 4-fold K_4

$$K_3 \xleftarrow{\pi} \mathcal{M}_7 \xleftarrow{\mathfrak{H}^{-1}} K_4 \xhookrightarrow{\mathcal{A}} \mathbb{V}_q$$

Projection

Inclusion into a higher dimensional algebraic variety

The N=8 case with no singularity

$$\mathbb{CP}^3 \xleftarrow{\pi} \mathbb{S}^7 \overset{\mathcal{C}one}{\hookrightarrow} \mathbb{C}^4 \overset{\mathcal{A}=\mathrm{Id}}{\hookrightarrow} \mathbb{C}^4$$

The near horizon limit produces the standard solution of D=11 SUGRA

$$\mathrm{AdS}_4 \times \mathbb{S}^7$$

This leads to the isometry group Osp(8|4) and to a free superconformal field theory on the M2 brane world volume, namely the Dirac singleton of Osp(8|4), with 8 bosons + 8 fermion degrees of freedom. The Kaluza Klein states are organized into short supermultiplets of Osp(8|4) that can be derived with purely group theoretical techniques.

The singular orbifold cases

Using the Hopf fibration of the 7-sphere

$$\pi \quad : \quad \mathbb{S}^7 \to \mathbb{CP}^3$$
$$\forall y \in \mathbb{CP}^3 \quad : \quad \pi^{-1}(y) \sim \mathbb{S}^1$$

We have

$$\frac{\mathbb{CP}^3}{\Gamma} \xleftarrow{\pi} \frac{\mathbb{S}^7}{\Gamma} \overset{Cone}{\hookrightarrow} \frac{\mathbb{C}^4}{\Gamma} \overset{\mathcal{A}=?}{\hookrightarrow} \quad ?$$

Where Γ is a finite subgroup of SU(4) with a linear holomorphic action on **C**^{4}

- ***We distinguish three cases***

Three cases

- A)

$$\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SU}(2)_{\mathrm{I}} \otimes \mathrm{SU}(2)_{\mathrm{II}} \subset \mathrm{SU}(4)$$

HyperKaehler quotient, N=4 susy in d=3 (McKay corr.)

$$\frac{\mathbb{C}^4}{\Gamma} \simeq \mathbb{C}^2 \times \left(\frac{\mathbb{C}^2}{\Gamma} \leftarrow \mathcal{M}_\zeta \right)$$

- B) Resolution of Kleinian singularities à la Kronheimer

$$\Gamma \subset \mathrm{SU}(3) \subset \mathrm{SU}(4)$$

Kaehler quotient, N=2 susy in d=3

$$\frac{\mathbb{C}^4}{\Gamma} \simeq \mathbb{C} \times \left(\frac{\mathbb{C}^3}{\Gamma} \leftarrow \mathcal{M}_\zeta \right)$$

- C) Generalized Kronheimer construction and McKay corr.

$$\Gamma \subset \mathrm{SU}(4) \quad \mathbf{4} = \mathrm{irred}$$

Very little is known so far.

Phys. ↔ Math. 1 ↔ 1 map

There is a one-to-one map between the field-content and the interaction structure of a $D = 3$, $\mathcal{N} = 2$ Chern-Simons gauge theory and the generalized Kronheimer algorithm of solving quotient singularities $\mathbb{C}^3/\Gamma$ via a Kähler quotient based on the McKay correspondence. All items on both sides of the one-to-one correspondence are completely determined by the structure of the finite group Γ and by its specific embedding into SU(3).

The map

Geometry

- S_Γ =$Hom_\Gamma(Q\times R,R)$ linear data, $\dim_C(S_\Gamma)$ =3 $|\Gamma|$
- G_Γ =quiver group (see later). F_Γ is the maximal compact subgroup thereof.
- The dimension is dim F_Γ = $|\Gamma|$ -1
- The moment map $\mu : S_\Gamma \to \mathbf{F}_\Gamma^*$ defines $|\Gamma|$ -1 functions $\mathcal{P}_I(q)$ that enter the Kaehler quotient construction
- one has to lift to level ζ_I>0 the moment maps associated with the center
- One needs a quadratic constraint $p\wedge p = 0$ that cuts a locus $V_{|\Gamma|+2}$ of dimension $|\Gamma|$+2:
- The Kaeher quotient of $V_{|\Gamma+2|}$ with respect to F_Γ is the minimal crepant resolution $M_\zeta \to C^3/\Gamma$

Chern Simons Gauge Theory

- S_Γ = Kaehler manifold of the Wess-Zumino multiplets (flat).
- F_Γ is the gauge group of the CS theory
- The dimension is dim F_Γ = $|\Gamma|$ -1
- The functions $\mathcal{P}_I(q)$ define the D terms and enter the formula for the scalar potential
- The level parameters ζ_I are the Fayet Iliopoulos parameters
- The equation $p\wedge p$=0 defines a universal cubic superpotential W_Γ
- The smooth manifold M_ζ is the space of vacua of the gauge theory

A flash of the Chern Simons super gauge theory

$$
\begin{aligned}
\mathscr{L}_{CSoff} \;=\; & -\alpha\,\mathrm{Tr}\left(\mathfrak{F}\wedge\mathscr{A}+\frac{2}{3}\mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A}\right)+\left(\frac{1}{2}g_{ij^\star}\,\Pi^{m|i}\,\nabla\bar{z}^{j^\star}+\overline{\Pi}^{m|j^\star}\,\nabla z^i\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
& -\frac{1}{6}\,g_{ij^\star}\,\Pi^{m|i}\,\overline{\Pi}^{m|j^\star}\,e^r\wedge e^s\wedge e^t\,\varepsilon_{rst}\\
& +\mathrm{i}\frac{1}{2}g_{ij^\star}\left(\overline{\chi}^{j^\star}\,\gamma^m\,\nabla\chi^i+\overline{\chi}^i_c\,\gamma^m\,\nabla\chi^{i^\star}_c\right)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
& \Bigg(-\frac{1}{3}M^\Lambda\left(\partial_i k^j_\Lambda\,g_{j\ell^\star}\overline{\chi}^{\ell^\star}\,\chi^i+\partial_{i^\star}k^{j^\star}_\Lambda\,g_{j\ell^\star}\overline{\chi}^\ell_c\,\chi^{i^\star}_c\right)+\frac{\alpha}{3}\left(\overline{\lambda}^\Lambda\,\lambda^\Sigma+\overline{\lambda}^\Lambda_c\,\lambda^\Sigma_c\right)\kappa_{\Lambda\Sigma}\\
& +\mathrm{i}\,\frac{1}{3}\left(\overline{\chi}^{j^\star}_c\,\lambda^\Lambda\,k^i_\Lambda-\overline{\chi}^i_c\,\lambda^\Lambda\,k^{j^\star}_\Lambda\right)g_{ij^\star}\\
& +\frac{1}{6}\left(\partial_i\partial_j\mathscr{W}\,\overline{\chi}^i_c\,\chi^j+\partial_{i^\star}\partial_{j^\star}\overline{\mathscr{W}}\,\overline{\chi}^{i^\star}\,\chi^{j^\star}_c\right)\Bigg)\wedge e^n\wedge e^p\,\varepsilon_{mnp}\\
& -V\left(M,D,\mathscr{H},z,\bar{z}\right)\varepsilon_{mnp}\,e^m\wedge e^n\wedge e^p
\end{aligned}
$$

$\left\{\mathcal{A}^\Lambda_\mu,\ \lambda^\Lambda,\ \lambda^\Lambda_c,\ M^\Lambda,\ D^\Lambda\right\}$ = gauge multiplets

$\left\{z^i,\ \chi^i,\ \mathcal{H}^i\right\}$ = Wess Zumino multiplets

General Form derived first in 1999 by D. Fabbri, P. Fré, L. Gualtieri, P. Termonia hepth/ 9905134 .
Mechanism of integration of the gauge multiplet that leads to the ABJM and Gaiotto forms derived first by M. Billò, D. Fabbri, P, Fré, P. Merlatti, A. Zaffaroni, in 2000 hepth/0005219
In present geometrical notation written by P. Fré and P.A. Grassi in arXiv:1705.00752 [hepth]

Elimination of the gauge multiplet fields

The scalars and the fermions in the gauge multiplet have algebraic field equations and can be integrated out , similarly for the WZ auxiliary fields.

$$M^{\Lambda} = \frac{1}{2\alpha} \kappa^{\Lambda\Sigma} \left(\mathcal{P}_{\Sigma} - \zeta_I \mathfrak{C}^I_{\Sigma} \right)$$

$$D^{\Lambda} = -\frac{1}{2\,\alpha^2} g_{ij^\star} \kappa^{\Lambda\Gamma} k^i_{\Gamma} k^{j^\star}_{\Sigma} \kappa^{\Sigma\Delta} \left(\mathcal{P}_{\Delta} - \zeta_I \mathfrak{C}^I_{\Delta} \right)$$

$$\mathcal{H}^i = g^{ij^\star} \partial_{j^\star} \overline{W} \quad ; \quad \overline{\mathcal{H}}^{j^\star} = g^{ij^\star} \partial_i W$$

Identification of non dynamical scalars.

$$\lambda^{\Lambda} = -\frac{1}{2\alpha} \kappa^{\Lambda\Sigma} g_{ij^\star} \chi^i k^{j^\star}_{\Sigma} \quad ; \quad \lambda^{\Lambda}_c = -\frac{1}{2\alpha} \kappa^{\Lambda\Sigma} g_{ij^\star} \chi^{j^\star} k^i_{\Sigma}$$

Identification of non dynamical gauginos

$k^i_{\Lambda}(z)$ = holomorphic Killing vector

$\kappa^{\Lambda\Sigma}$ = Killing metric

The final form of the scalar potential

$$V(z,\bar{z}) \;=\; \frac{1}{6}\left(\partial_i\mathcal{W}\,\partial_{j^\star}\overline{\mathcal{W}}\,g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma}\left(\mathcal{P}_\Lambda - \zeta_I\,\mathfrak{C}^I_\Lambda\right)\left(\mathcal{P}_\Sigma - \zeta_J\,\mathfrak{C}^J_\Sigma\right)\right)$$

$$\mathbf{m}^{\Lambda\Sigma}(z,\bar{z}) \;\equiv\; \frac{1}{4\alpha^2}\,\kappa^{\Lambda\Gamma}\,\kappa^{\Sigma\Delta}\,k^i_\Gamma\,k^{j^\star}_\Delta\,g_{ij^\star}$$

The manifold of extrema of the scalar potential coincides with the minimal crepant resolution of the singularity $\mathbb{C}^3/\Gamma$ according with the **generalized Kronheimer construction** based on the generalized **McKay correspondence**.
Indeed since the potential is a sum of squares the extrema are defined by

$$\partial_i\mathcal{W} \;=\; 0 \quad \Longleftrightarrow \quad \mathbf{p}\wedge\mathbf{p} = 0$$

$$\mathcal{P}_\Lambda \;=\; \zeta_I\,\mathfrak{C}^I_\Lambda$$

Furthermore gauge invariance implies that we have to consider only orbits of the gauge group and this completes the Kaehler quotient procedure.

The diagram of the smooth resolution in case A)

$$\mathcal{M}_7 \stackrel{\mathfrak{H}^{-1}}{\longleftarrow} \mathbb{C}^2 \times ALE_\Gamma \stackrel{\mathrm{Id}\times qK}{\longleftarrow} \mathbb{C}^2 \times \mathbb{V}_{|\Gamma|+1} \stackrel{\mathcal{A}_\mathcal{P}}{\hookrightarrow} \mathbb{C}^2 \times \mathbb{C}^{2|\Gamma|}$$

The map $\stackrel{\mathcal{A}_\mathcal{P}}{\hookrightarrow}$ denotes the inclusion map of the variety $\mathbb{V}_{|\Gamma|+1}$ in $\mathbb{C}^{2|\Gamma|}$

$$qK \quad : \quad \mathbb{V}_{|\Gamma|+1} \longrightarrow \mathbb{V}_{|\Gamma|+1} /\!/_K \mathcal{F}_{|\Gamma|-1} \simeq ALE_\Gamma$$

qK is the Kaehler quotient with respect to the gauge group.

Altogether we have a HyperKaehler quotient

$$ALE_\Gamma = \mathbb{C}^{2|\Gamma|} /\!/_{HK} \mathcal{F}_{|\Gamma|-1}$$

It is convenient to split the HK quotient into two steps in order to compare with the case C^3/Γ

The diagram of the smooth resolution in case B)

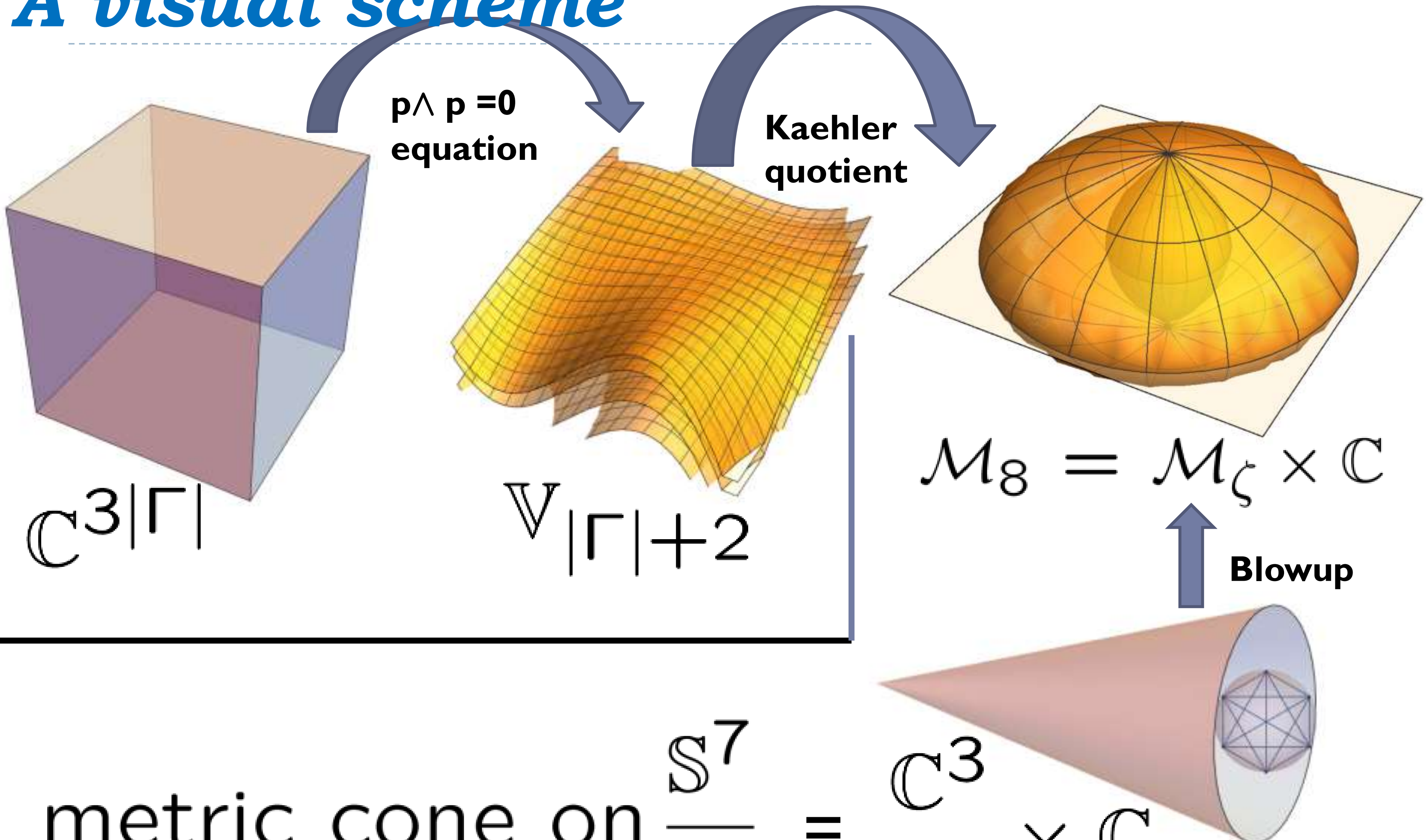
$$\mathcal{M}_7 \xleftarrow{\mathfrak{H}^{-1}} \mathbb{C} \times Y_\Gamma \xleftarrow{\mathrm{Id} \times qK} \mathbb{C} \times \mathbb{V}_{|\Gamma|+2} \xhookrightarrow{\mathrm{Id} \times \mathcal{A}_\mathcal{P}} \mathbb{C} \times \mathbb{C}^{3|\Gamma|}$$

The intermediate step, just as in case A) is a Kaehler quotient, yet the starting point variety $V_{|\Gamma|+2}$ has a different definition. From the physical viewpoint we have N=2 rather than N=4 susy and the superpotential is not defined by the holomorphic moment maps, this corresponds mathematically to the fixed equation $p \wedge p = 0$
that amount s to identifying $V_{|\Gamma|+2}$ with a certain orbit with respect to the quiver group G_Γ, actually the compactification of the gauge group $F_{\Gamma:}$

$$\mathbb{V}_{|\Gamma|+2} = \mathcal{D}_\Gamma \equiv \mathrm{Orbit}_{\mathcal{G}_\Gamma}(L_\Gamma)$$

We see later what the locus L_Γ is.

A visual scheme



$$\text{metric cone on } \frac{\mathbb{S}^7}{\Gamma} = \frac{\mathbb{C}^3}{\Gamma} \times \mathbb{C}$$

The McKay correspondence for $\frac{\mathbb{C}^2}{\Gamma}$

In finite group theory we have the decomposition of any rep. D into irreps D_μ

Let Q be the defining rep. of $\Gamma \subset SU(2)$

$$Q \otimes D_\mu = \bigoplus_{\nu=0}^{r} A_{\mu\nu} D_\nu$$

$$D = \bigoplus_{\mu=1}^{r} a_\mu D_\mu$$

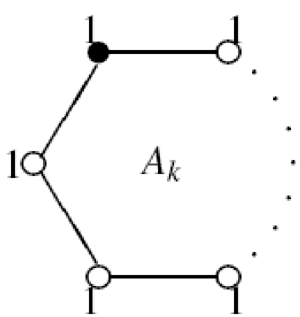
$$a_\mu = \frac{1}{g} \sum_{i=1}^{r} g_i \chi_i^{(D)} \chi_i^{(\mu)\star}$$

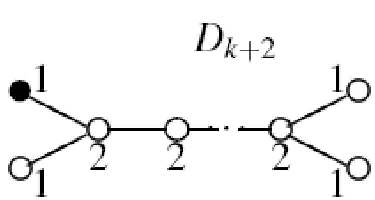
The isomorphic ADE classification of Kleinian groups Γ and semisimple Lie algebras is known.
The Coxeter numbers coincide with the dimensions of Γ irreps

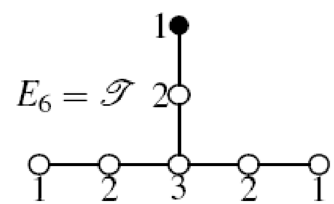
Miracolous properties the matrix

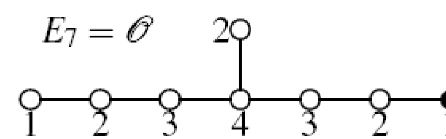
$$c_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}$$

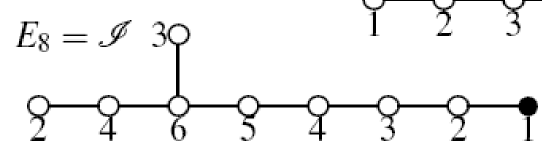
It is the extended Cartan matrix of the Lie algebras **A_k, D_{k+2}, E_6, E_7, E_8**











McKay corr. and the Kronheimer construction of ALE manifolds

Define a space $\mathcal{P}$ of pairs p= (A,B) of complex $|\Gamma| \times |\Gamma|$ matrices. Define the action of the group Γ on $\mathcal{P}$

$$\forall \gamma \in \Gamma \quad : \quad \begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \mathcal{Q}_\gamma \begin{pmatrix} R(\gamma)\, A\, R(\gamma^{-1}) \\ R(\gamma)\, B\, R(\gamma^{-1}) \end{pmatrix}$$

where R(γ) is the regular representation and $\mathcal{Q}_\gamma$ the defining representation.
In intrinsic notation

$$\mathcal{P} \simeq \mathrm{Hom}\,(R, \mathcal{Q} \otimes R)$$

Introduce the Γ invariant subspace of $\mathcal{P}$

$$\mathcal{S} \equiv \{p \in \mathcal{P} / \forall \gamma \in \Gamma, \gamma \cdot p = p\} = \mathrm{Hom}_\Gamma\,(R, \mathcal{Q} \otimes R)$$

The space $\mathcal{S}$ is a flat Kaehler manifold with **complex dimension 2|Γ|** which encodes (in Physics) the **Wess-Zumino multiplets** of the CS theory (actually) **hypermultiplets** since susy will be N=4.

Why the dimension of $\mathcal{S}$ is $2|\Gamma|$?

The answer is:

1. **McKay correspondence**
2. **Regular representation**
3. **Schur's Lemma**

$$\mathcal{S} = \bigoplus_{\mu,\nu} A_{\mu,\nu} \mathrm{Hom}(\mathbb{C}^{n_\mu}, \mathbb{C}^{n_\nu})$$

$$\dim_{\mathbb{C}} \left[\mathrm{Hom}_{\Gamma}(R, \mathcal{Q} \otimes R)\right] = 2|\Gamma|$$

Actually the space $\mathcal{S}$ is a flat HyperKaehler manifold with a triplet of Kaehler forms arranged into a quaternion

$$\Theta = \mathrm{Tr}(\mathrm{d}p^\dagger \wedge \mathrm{d}p) = \begin{pmatrix} \mathrm{i}\,\mathrm{K} & \mathrm{i}\overline{\Omega} \\ \mathrm{i}\,\Omega & -\mathrm{i}\,\mathrm{K} \end{pmatrix}$$

$$\mathbf{K} = -\mathrm{i}\left[\mathrm{Tr}\left(\mathrm{d}A^\dagger \wedge \mathrm{d}A\right) + \mathrm{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}B\right)\right] \equiv \mathrm{i} g_{\alpha\bar{\beta}}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^{\bar{\beta}}$$

$$ds^2 = g_{\alpha\bar{\beta}}\, \mathrm{d}q^\alpha \otimes \mathrm{d}q^{\bar{\beta}}$$

$$\mathbf{\Omega} = 2\mathrm{Tr}(\mathrm{d}A \wedge \mathrm{d}B) \equiv \Omega_{\alpha\beta}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta$$

This allows to perform a HyperKaehler quotient with respect to a ***suitable gauge group*** $\mathcal{F}_\Gamma$

The *gauge group* and the *quiver group*

$$\mathcal{F}_\Gamma = \bigotimes_{\mu=1}^{r+1} \mathsf{U}(\mathsf{n}_\mu) \bigcap \mathsf{SU}(|\Gamma|)$$ ***gauge group***

$$\mathcal{G}_\Gamma = \bigotimes_{\mu=1}^{r+1} \mathsf{GL}(\mathsf{n}_\mu, \mathbb{C}) \bigcap \mathsf{SL}(|\Gamma|, \mathbb{C})$$ ***quiver group***

The gauge group is the maximal compact subgroup of the quiver group, the latter being the complexification of the former. The real dimension of the gauge group is $|\Gamma|$-1, the complex dimension of the quiver group is the same. We have the ***triholomorphic moment map***, well known in supersymmetric gauge theories (D-terms)

$$\mu \; : \; \mathcal{S}_\Gamma \longrightarrow \mathbb{R}^3 \otimes \mathbb{F}_\Gamma^\star$$

dual of the gauge Lie algebra

$$\begin{aligned} \mu_3(p) &= -i\left([A, A^\dagger] + [B, B^\dagger]\right) \\ \mu_+(p) &= ([A, B]) \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_A^3 &= \mathsf{Tr}\left(\mu_3(p)\, f_A\right) \\ \mathfrak{P}_A^+ &= \mathsf{Tr}\left(\mu_+(p)\, f_A\right) \end{aligned}$$

real moment maps

holomorphic moment maps

The ***Discreet Charm*** of the integer ***r***

The integer ***r*** counts several distinct things at the same time

1. The number of non trivial **irreps** of Γ.
2. The number of non trivial **conjugacy classes** of Γ.
3. The dimension of the center $\mathfrak{z}[\mathbb{F}_\Gamma]$ of the gauge Lie algebra.
4. Hence the number of **Fayet Iliopoulos parameters** in the CS supergauge theory.
5. As we will see also the number of **tautological holomorphic bundles** on the resolved variety: $M_\zeta \to \mathbb{C}^n/\Gamma$ (n=2,3)
6. In the case n=2 (ADE) the **rank** of the semisimple Lie algebra corresponding to Γ.

The resolved smooth manifold ALE_Γ is obtained as the **HyperKaeler quotient** of $\mathcal{S}_\Gamma$ by $\mathcal{F}_\Gamma$

$$\mathcal{M}_\zeta \equiv \mu^{-1}(\vec{\zeta}) \,/\!/\, \mathcal{F}_\Gamma$$

where

$$\vec{\zeta} \in \mathbb{R}^3 \otimes \mathfrak{z}\left[\mathbb{F}_\Gamma\right]^\star$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 1°

Next let $\Gamma \subset$ **SU(n).** We have a generalized McKay correspondence

$$\mathcal{Q} \otimes \mathrm{D}_i = \bigoplus_{j=1}^{r+1} \mathcal{A}_{ij}\, \mathrm{D}_j$$

$$\bar{c}_{ij} = n\,\delta_{ij} - \mathcal{A}_{ij}$$

generalized extended Cartan matrix

$$\mathbf{n} \equiv \{1, n_1, \ldots, n_r\}$$

vector of irrep dimensions

$$\bar{c}.\mathbf{n} = 0$$

fundamental property

For n=3 we introduce a space $\mathcal{P}_\Gamma$ of triplets of $|\Gamma|\times|\Gamma|$ matrices

$$p \in \mathcal{P}_\Gamma \equiv \mathrm{Hom}\,(R, \mathcal{Q} \otimes R) \Rightarrow p = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 2°

Similarly we define the invariant subspace

$$\mathcal{S}_\Gamma \equiv \mathrm{Hom}_\Gamma(R, Q \otimes R) = \{p \in \mathcal{P}_\Gamma / \forall \gamma \in \Gamma, \gamma \cdot p = p\}$$

where the group action is

$$\forall \gamma \in \Gamma : \quad \gamma \cdot p \equiv \mathcal{Q}(\gamma) \begin{pmatrix} R(\gamma)\, A\, R(\gamma^{-1}) \\ R(\gamma)\, B\, R(\gamma^{-1}) \\ R(\gamma)\, C\, R(\gamma^{-1}) \end{pmatrix}$$

Because of the McKay relation we have

$$\mathcal{S}_\Gamma = \bigoplus_{i,j} A_{i,j} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

$$\Downarrow$$

$$\dim_{\mathbb{C}} \mathcal{S}_\Gamma = 3 \sum_i n_i^2 = 3|\Gamma|$$

The space $\mathcal{S}_\Gamma$ is a flat Kaehler manifold of dimension 3|Γ|. It accomodates the WZ multiplets of the N=2 CS gauge theory. So there are no holomorphic moment maps but we can have a superpotential $\mathcal{W}$ the vanishing of whose derivatives provides holomorphic constraints.

n=3 generalization of the McKay corr. and Kronheimer construction STEP 3°

How can we step down from **3|Γ|** complex dimensions to **3-dimensions**?
The gauge group $\mathcal{F}_\Gamma$ has **|Γ|-1 generators** and the corresponding Kaehler quotient kills |Γ|-1 complex parameters. Hence the starting point should be a **variety with complex dimensions |Γ| +2.**

Question: *what is the analogue of holomorphic moment map equation?*

Answer: it is $\mathbf{p} \wedge \mathbf{p} = 0$

$$0 = \epsilon^{ijk} \mathbf{p}_i \cdot \mathbf{p}_j$$
$$\Updownarrow$$
$$0 = [A, B] = [B, C] = [C, A]$$

The general solution to this constraint is given by a variety $\mathbf{V}_{|\Gamma|+2}$ that can be seen as the quiver group orbit of a special 3-dimensional locus

$$\mathbb{V}_{|\Gamma|+2} = \mathcal{D}_\Gamma \equiv \mathrm{Orbit}_{\mathcal{G}_\Gamma}(L_\Gamma)$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 4°

$$\mathcal{S}_\Gamma \supset L_\Gamma \equiv \left\{ \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \in \mathcal{S}_\Gamma \;\middle|\; A_0, B_0, C_0 \text{ diagonal in natural basis of R} \right\}$$

The locus L_Γ is easily seen to be 3-dimensional and we have

$$\mu^{-1}(0) = \text{Orbit}_{\mathcal{F}_\Gamma}(L_\Gamma)$$

Hence L_Γ describes the singular orbifold $\mathbb{C}^3/\Gamma$

Introducing the orthogonal decomposition

$$\mathbb{G}_\Gamma = \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma$$

$$[\mathbb{F}_\Gamma, \mathbb{F}_\Gamma] \subset \mathbb{F}_\Gamma \quad ; \quad [\mathbb{F}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{K}_\Gamma \quad ; \quad [\mathbb{K}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{F}_\Gamma$$

$$\mathcal{D}_\Gamma = \text{Orbit}_{\mathcal{F}_\Gamma}(\exp[\mathbb{K}_\Gamma] \cdot L_\Gamma)$$

$$\mu^{-1}(\zeta) /\!/ \mathcal{F}_\Gamma = \left\{ Z \in \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \;\middle\|\; \begin{array}{ll} \mu_I(Z) = 0 & \text{if } f_I \notin \mathfrak{z} \\ \mu_I(Z) = \zeta_I & \text{if } f_I \in \mathfrak{z} \end{array} \right\}$$

Other characterizations of the space $\mathcal{D}_{\Gamma}$

At the moment we are studying other possible characterizations of this variety as a quotient of C^n with respect to some $C^* \times \ldots \times C^*$

It is an open problem that may lead to new visions

The moment map equation

The solution of the singularity resolution problem is finally reduced to an algebraic equation for the coset element

$$\mathcal{V} = \exp\left[\mathbf{\Phi}\right] \quad ; \quad \mathbf{\Phi} \in \mathbb{K}_{\Gamma}$$

$$\mathcal{V} = \begin{pmatrix} \mathfrak{H}_0 & 0 & 0 & \dots & 0 \\ 0 & \mathfrak{H}_1 \otimes \mathbf{1}_{n_1 \times n_1} & 0 & \dots & \vdots \\ 0 & 0 & \mathfrak{H}_2 \otimes \mathbf{1}_{n_2 \times n_2} & \dots & \vdots \\ \vdots & \dots & \dots & \dots\dots & \vdots \\ \vdots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & \mathfrak{H}_r \otimes \mathbf{1}_{n_r \times n_r} \end{pmatrix}$$

Such that

$$\mu\left(\mathcal{V} \cdot L_{\Gamma}\right) = \zeta$$

Typically that above is a system of algebraic equations of higher order. In few cases one can reduce it to order 4°, 3° or 2° obtaining solutions by radicals.

The tautological bundles

From the coset element $\mathcal{V}$ we extract a hermitian matrix

$$\mathcal{H} \equiv \begin{pmatrix} \mathfrak{H}_1 & 0 & \dots & \dots & 0 \\ 0 & \mathfrak{H}_2 & \dots & \dots & \vdots \\ \vdots & \dots\dots & \dots\dots & \dots & \vdots \\ \vdots & \dots\dots & \dots & \mathfrak{H}_{r-1} & 0 \\ 0 & \dots\dots & \dots & 0 & \mathfrak{H}_r \end{pmatrix}$$

that is the fiber metric on the direct sum

$$\mathcal{R} = \bigoplus_{i=1}^{r} \mathcal{R}_i$$

of **r** tautological bundles that, by construction, are holomorphic vector bundles with **rank** equal to the dimensions $\mathbf{n_i}$ of the **r** irreps of Γ:

$$\mathcal{R}_i \xrightarrow{\pi} \mathcal{M}_\zeta \quad ; \quad \forall p \in \mathcal{M}_\zeta \quad : \quad \pi^{-1}(p) \approx \mathbb{C}^{n_i}$$

Provided we are able to solve the moment map equation we can evaluate the first Chern classes of these bundles

$$\omega_i^{(1,1)} \equiv c_1(\mathcal{R}_i) = \frac{i}{2\pi} \overline{\partial} \partial \log \left[\mathrm{Det} \left(\mathfrak{H}_i \right) \right]$$

One simple master example

$$\frac{\mathbb{C}^3}{\mathbb{Z}_3}$$ generated by $$Y = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix}$$ $\xi^3 = 1$

$$a - \text{vectors} = \left\{\{0,0,0\}, \frac{1}{3}\{1,1,1\}, \frac{1}{3}\{2,2,2\}\right\}$$

Hence we conclude that the Hodge numbers of the resolved variety should be $h^{(0,0)} = 1$; $h^{(1,1)} = 1$; $h^{(2,2)} = 1$.

Following the generalized Kronheimer construction one arrives at the following system of algebraic equations for the entries of the H-matrix (moment map equation)

$$\left\{\frac{\Sigma\left(-1+\Upsilon_1^3\right)}{\Upsilon_1\Upsilon_2}, \frac{\Sigma\left(-1+\Upsilon_2^3\right)}{\Upsilon_1\Upsilon_2}\right\} = \{\zeta_1, \zeta_2\}$$

where

$$\Sigma = \sum_{i=1}^{3} |z_i|^2$$

Thanks Cardano & T artaglia!

The moment map equation is solvable by radicals!

We can explicitly calculate the $\omega^{(1,1)}{}_{1,2}$ forms

$$\omega^{(1,1)}_{1,2} = \frac{i}{2\pi}\left(\frac{d}{\mathrm{d}\Sigma}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] d\bar{z}^i \wedge \mathrm{d}z^i + \frac{d^2}{\mathrm{d}\Sigma^2}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] z^j \bar{z}^i \mathrm{d}z^i \wedge d\bar{z}^j\right)$$

Introducing the intersection integral

$$\mathcal{I}_{abc} = \int_{\mathcal{M}} \omega_a^{(1,1)} \wedge \omega_b^{(1,1)} \wedge \omega_b^{(1,1)}$$

We find

$$\begin{array}{lcl} (\zeta_1 > 0, \zeta_2 = 0) & : & \mathscr{I}_{111} = \frac{1}{8} \\ (\zeta_1 = 0, \zeta_2 \geq 0) & : & \mathscr{I}_{111} = 0 \\ (\zeta_1 > 0, \zeta_2 > 0) & : & \mathscr{I}_{111} = 1 \end{array}$$

$$\begin{array}{lcl} (\zeta_1 > 0, \zeta_2 = 0) & : & \mathscr{I}_{211} = 0 \\ (\zeta_1 = 0, \zeta_2 \geq 0) & : & \mathscr{I}_{211} = 0 \\ (\zeta_1 > 0, \zeta_2 > 0) & : & \mathscr{I}_{211} = 1 \end{array}$$

Another case under investigation:

$$\frac{\mathbb{C}^3}{\mathbb{Z}_4}$$

Generator of $\mathbf{Z}_4$

$$Y = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The moment map equations

$$\begin{pmatrix} -\frac{(X_1^2-X_3^2)(X_1X_3(\Delta_1^2+\Delta_2^2)+(1+X_2^2)\Delta_3^2)}{X_1X_2X_3} \\ \frac{(X_2+X_2^3-X_1X_3(X_1^2+X_3^2))(\Delta_1^2+\Delta_2^2)}{X_1X_2X_3} \\ -\frac{(-1+X_2^2)(X_2(\Delta_1^2+\Delta_2^2)+(X_1^2+X_3^2)\Delta_3^2)}{X_1X_2X_3} \end{pmatrix} = \begin{pmatrix} \zeta_1-\zeta_3 \\ \zeta_1-\zeta_2+\zeta_3 \\ \zeta_2 \end{pmatrix}$$

Predictions from Ito-Reid Theorem

age-vectors

$$c_1 = \frac{1}{4}\{1,1,2\}$$ **junior compact**

$$c_2 = \frac{1}{4}\{2,2,0\}$$ **junior non compact**

$$c_3 = \frac{1}{4}\{3,3,2\}$$ **senior**

Poincaré duality

since we have a compact support (1,1)-cocycle there must be also a (2,2)-cocycle

Here we have a complete illustration. $\mathbf{Z}_4$ has 3 non trivial irreps hence there are three tautological bundles and three $\omega^{1,1}$ closed forms. Yet we expect only two 2-cycles in homology since we have only 2 junior classes. In the ***correspondence line-bundles divisors*** only ***one compact divisor*** and ***one non compact one***. There is a linear relation between the cohomology classes of the three $\omega^{1,1}$ closed forms.

Let us retrieve these predictions from the toric description.

The lattice of invariants

$$M \equiv \left\{ \mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = \bigoplus_{i=1}^{3} \mathrm{m}^i \, \epsilon_i^\vee , \quad \mathrm{m}^i \in \mathbb{Z} \right\}$$

$$\begin{array}{lcl} \epsilon_1^\vee & = & \{1,-1,0\} & \Leftrightarrow & \mathcal{I}_1 \\ \epsilon_2^\vee & = & \{0,2,-1\} & \Leftrightarrow & \mathcal{I}_2 \\ \epsilon_3^\vee & = & \{0,0,2\} & \Leftrightarrow & \mathcal{I}_2 \end{array}$$

Basis of invariant Laurent monomials

$$\mathcal{I}_1 = x\,y^{-1}, \quad \mathcal{I}_2 = y^2\,z^{-1}, \quad \mathcal{I}_3 = z^2$$

The dual lattice N

$$N = M^\vee \equiv \left\{ \mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = \bigoplus_{i=1}^{3} \mathfrak{n}_i \, \epsilon^i , \quad \mathfrak{n}_i \in \mathbb{Z} \right\}$$

$$\epsilon_i^\vee \cdot \epsilon^j = \delta_i^j$$

$$\epsilon^1 = \{1,0,0\}, \quad \epsilon^2 = \tfrac{1}{4}\{2,2,0\}, \quad \epsilon^3 = \tfrac{1}{4}\{1,1,2\}$$

Age vectors of the two junior classes

This is not a fortuitous coincidence. Because ....

Toric description

A complex n-dimensional toric variety is described by a fan of rays r_i in R^n that define a collection of maximal convex cones σ_i and each cone σ is associated with an open chart U_σ forming the atlas that covers the variety.

In each chart U_σ the coordinate ring is given by

$$X_{\sigma_i} = \text{Spec}\, \mathbb{C}\left[\sigma_i^\vee \bigcap M\right]$$

This is an abstract notation to say something simple. We simply have a prescription how to write in each chart the local coordinates *u,v,w* in terms of invariant indipendent monomials of the original coordinates of $\mathbb{C}^3$

$$\sigma^\vee = \text{Cone}\,(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$$

Local Coordinates in chart U_σ

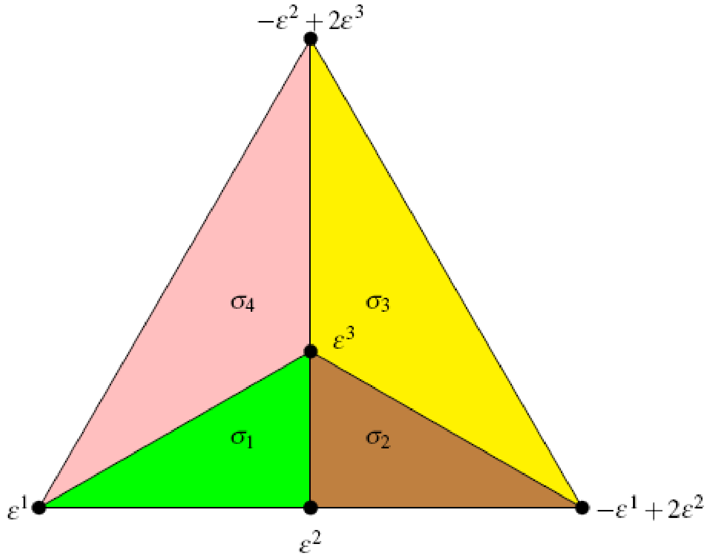
$$u = x^{\mathfrak{a}_1} y^{\mathfrak{a}_2} z^{\mathfrak{a}_3}$$
$$v = x^{\mathfrak{b}_1} y^{\mathfrak{b}_2} z^{\mathfrak{b}_3}$$
$$w = x^{\mathfrak{c}_1} y^{\mathfrak{c}_2} z^{\mathfrak{c}_3}$$

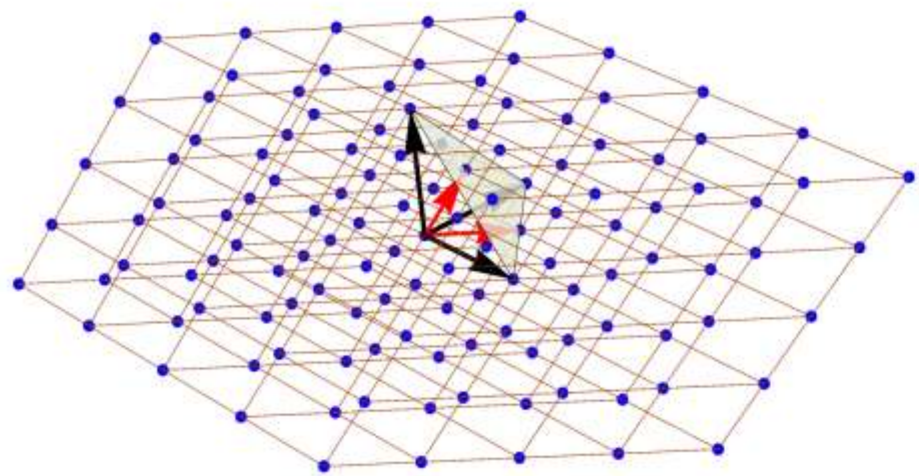
Procedure

- Define the invariant lattice M
- Derive the direct lattice N as the dual of N
- Introduce the fan Φ of rays and the cones σ_i
- Consider the maximal cones σ_i contained in the fan Φ
- Construct the dual cones $\sigma^{\vee}{}_i$ and the associated open charts. The transition functions are given and one has the atlas covering the toric variety

QUESTION: ***What is the fan of the resolution?***
ANSWER: ***One starts from the fan of C^3 and adds..... see next slide***

The fan and the cones for $\mathbb{C}^3/\mathbb{Z}_4$





$$\mathbf{e}_1 = \{1,0,0\}, \quad \mathbf{e}_2 = \{0,1,0\}, \quad \mathbf{e}_3 = \{0,0,1\}$$

$$\mathbf{e}_1 = \epsilon^1, \quad \mathbf{e}_2 = -\epsilon^1 + 2\,\epsilon^2, \quad \mathbf{e}_3 = -\epsilon^2 + 2\,\epsilon^3$$

Which points of the N lattice lie on the face whose end points are $\mathbf{e}_{1,2,3}$?
The corresponding rays have to be added to the fan. They are always the **junior classes age vectors!** **MIRACLE**

Exceptional Divisor

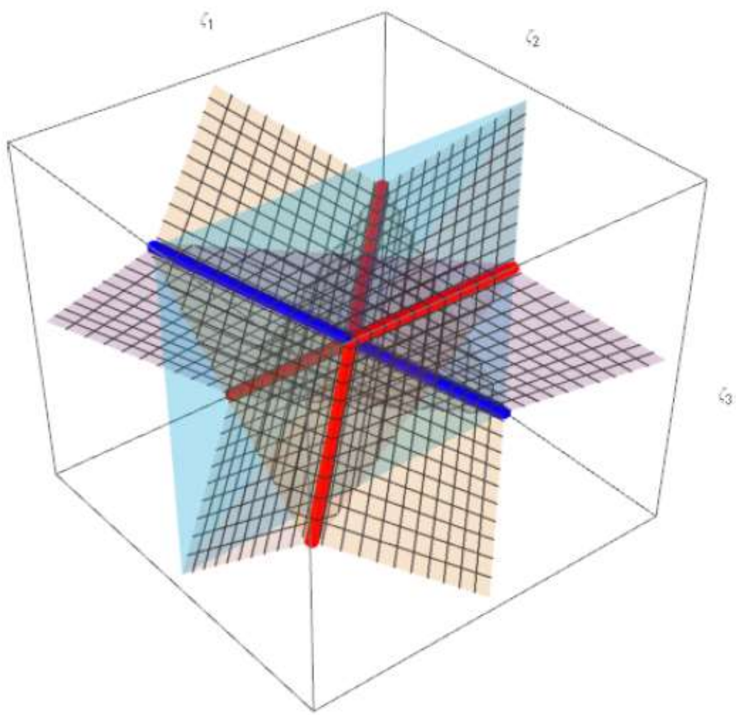
The final outcome of the construction is the following atlas of open charts

$$\begin{array}{llll} \text{Chart } X_{\sigma_1} & x \to u\sqrt{v}\,\sqrt[4]{w} & , \; y \to \sqrt{v}\,\sqrt[4]{w} & , \; z \to \sqrt{w} \\ \text{Chart } X_{\sigma_2} & x \to \sqrt{v}\,\sqrt[4]{w} & , \; y \to u\sqrt{v}\,\sqrt[4]{w} & , \; z \to \sqrt{w} \\ \text{Chart } X_{\sigma_3} & x \to \sqrt[4]{w} & , \; y \to u\,\sqrt[4]{w} & , \; z \to v\sqrt{w} \\ \text{Chart } X_{\sigma_4} & x \to u\,\sqrt[4]{v} & , \; y \to \sqrt[4]{v} & , \; z \to \sqrt{vw} \end{array}$$

In charts 1,2,3 the locus *w*=0 is the blowup of the singular point x=y=z=0. It is the compact exceptional divisor.
In charts 4 same locus is given by *v*=0

We are studying the topology of the exceptional divisor and we have determined that is the second Hirzebruch surface in $P^2 \times P^1$

The Chamber Structure



In the bulk the Kaehler quotient always gives the complete smooth resolution. On the walls there are degeneration or partial resolutions. In any case crossing a wall the periods of the tautological bundles change

The blue line is the degeneration EH
The red lines are the degenerations Cardano.

The degeneration Y3

The degeneration Y3 which is the model of what happens on the walls is the canonical line bundle over the weighted projective space P[1,1,2]. This space is singular. Hence it is a partial resolution.

The degeneration EH

This space is the smooth space C x Eguchi Hanson

The resolved variety Y

Y is topologically and analytically the total space of the canonical line bundle over the second Hirzebruch surface

Let us give some details about the geometry of the second Hirzebruch surface $\mathbb{F}_2$, which appears as the compact component of the exceptional divisor of the resolution $Y \to \mathbb{C}^3/Z_4$.

Let (U,V) be homogeneous coordinates on $\mathbb{P}^1$ and (X,Y,Z) homogeneous coordinates of $\mathbb{P}^2$.

Definition 5.1. *The n-th Hirzebruch surface $\mathbb{F}_n$ is defined as the locus cut out in $\mathbb{P}^1 \times \mathbb{P}^2$ by the following equation of degree $n+1$:*

$$0 = \mathscr{P}_n(U,V,X,Y,Z) = X\,U^n - Y\,V^n \tag{5.1}$$

Conclusions

- The final goal is to match the Kronheimer like construction (= gauge theory model) with the algebraic constructions of the resolved variety, in particular deriving the exceptional divisors
- The compact exceptional divisors are where branes can be wrapped in M2 and D3 applications.
- So far the toric description was helpful yet the goal is to consider also non abelian Γ cases. Ito-Reid theorem applies also to them.
- In perspective we have new ABJM Chern Simons gauge theories and resolved fractional brane gauge theories in D=4.
- The largest possible Γ is the simple group L_{168}. The resolution $\mathbf{C}^3 \backslash L_{168}$ was constructed 22 years ago by Markushevich. It is a hypersurface in $\mathbf{C}^4$