Resolutions à la Kronheimer of orbifold singularities, McKay quivers for Gauge Theories on D3 branes, and Ricci flat metrics on the resolved three-folds

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Pietro G. Fré
University of Torino

Albert Einstein Institute Potsdam
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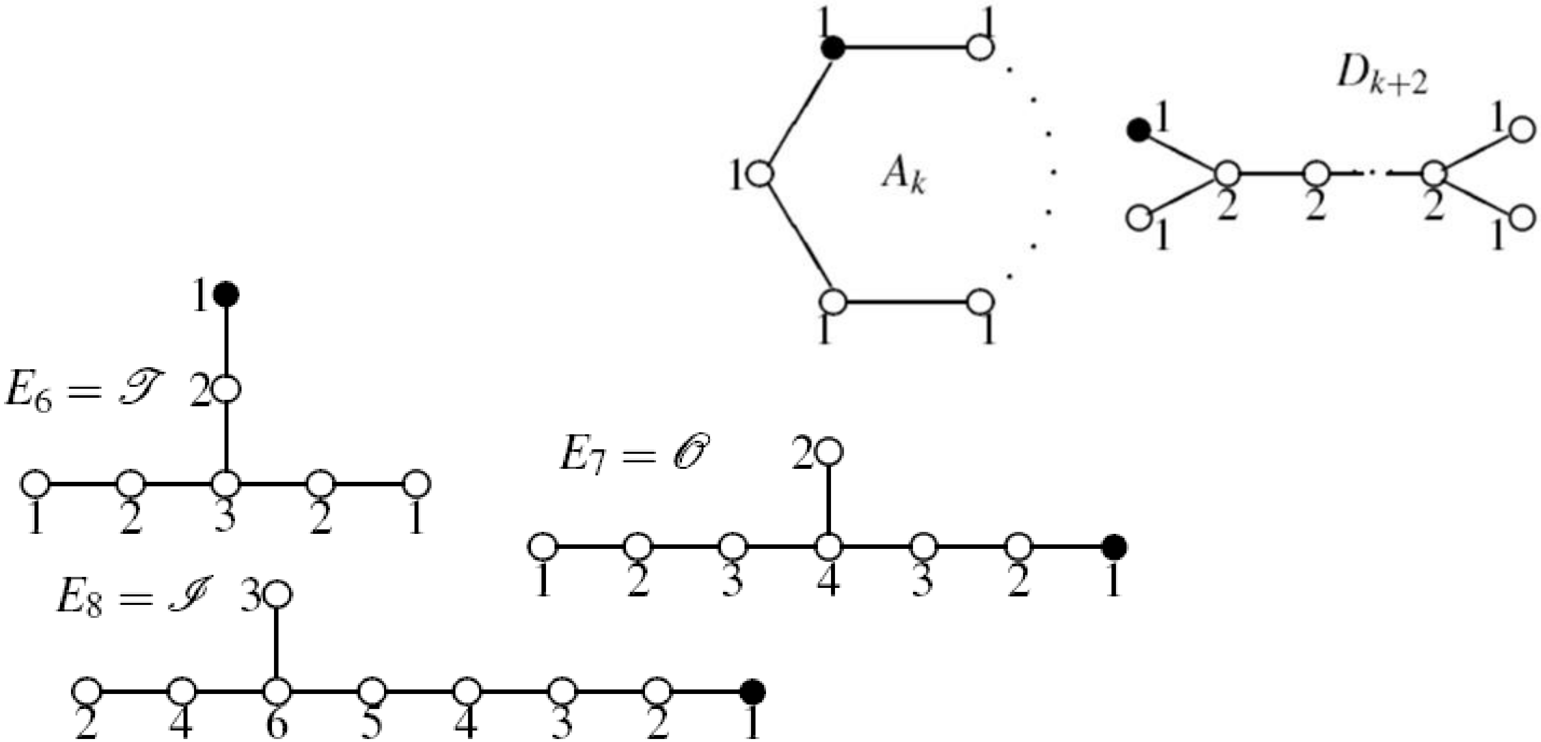
FIRST LECTURE

From the Kronheimer case C^2/Γ

To the generalized Kronheimer construction C^3/Γ



Series of papers

1) U. Bruzzo, A. Fino, and P. Fré, *The Kähler quotient resolution of $\mathbb{C}^3/\Gamma$ singularities, the McKay correspondence and D=3 $\mathcal{N}=2$ Chern-Simons gauge theories*, Commun. Math. Phys., 365 (2019), pp. 93–214.

2) U. Bruzzo, A. Fino, P. Fré, P. A. Grassi, and D. Markushevich, *Crepant resolutions of $\mathbb{C}^3/\mathbb{Z}_4$ and the generalized Kronheimer construction (in view of the gauge/gravity correspondence)*, J. Geom. Phys., 145 (2019), pp. 103467, 50.

3) M. Bianchi, U. Bruzzo, P. Fré, and D. Martelli, *Resolution à la Kronheimer of $\mathbb{C}^3/\Gamma$ singularities and the Monge-Ampère equation for Ricci-flat Kähler metrics in view of D3-brane solutions of supergravity*, Lett. Math. Phys., 111 (2021).

4) **arXiv:2211.10353** math-ph, hep-th, math.DG
D3-brane supergravity solutions from Ricci-flat metrics on canonical bundles of Kähler-Einstein surfaces
Ugo Bruzzo, Pietro Fré, Umar Shahzad, Mario Trigiante

Setting the stage

$\mathscr{A}$) **Holography and the Gauge/Gravity Correspondence**. This is the idea that a quantum field theory living on the boundary $\partial \mathscr{M}_{ST}$ of some multi-dimensional space time, for instance the gauge theory describing all the non gravitational interactions that lives on the four-dimensional boundary of five-dimensional anti de Sitter space AdS_5 might be *"solved"* by means of classical geometrical calculations in the bulk $\mathscr{M}_{ST}$.

$\mathscr{B}$) **The breaking of conformal invariance of the gauge theory by the resolution of orbifold singularities.** This is the development within Supergravity and String Theory of the consequences of the following chain of results piled up in the last four decades:

1. a way to break supersymmetry or other continuous symmetries is that of considering as substratum of exact solutions of (Super)-Gravity field equations, flat manifolds $\mathscr{M}_{flat}$ quotiened by the action of a discrete group Γ, generically named orbifolds $\mathscr{M}_{flat}/\Gamma$.
2. The space $\mathscr{M}_{flat}/\Gamma$ has singularities in the fixed points for the action of Γ on $\mathscr{M}_{flat}$ and there are relevant operators that can deform the orbifold solution to one on a smooth manifold $\widetilde{\mathscr{M}}_{smooth}$ that, via the exceptional divisors introduced by the blowup morphism, develops non trivial homology and cohomology.
3. The size of such new homology cycles are dimensionful parameters that break conformal invariance and give rise to more realistic holographic pairs *gauge-theory/gravitational solution*.

Group Structures

Tracing back physical theories to group structures. This is the guiding principle, physical and philosophical that aims at reducing the Laws of Nature to Symmetry Principles. In its declination in the context of holography and orbifolds one would like to develop a *robust conceptual setup* in which the dual pairs and all of their aspects can be traced back to *group structures*.

The robust conceptual setup

The *robust conceptual setup* that provides a rich play-ground for the theoretical physicist's aspirations concisely described about was created by mathematicians at the end of 1980.s In particular by Peter Kronheimer in with an essential input from the genial discoveries of John McKay. It is the Kronheimer setup for the resolution of the $\mathbb{C}^2/\Gamma$ singularities where $\Gamma \subset \mathrm{SU}(2)$. The resolution produces the so named ALE-manifolds

Phys. ⟺ Math. 1 ⟺ 1 map

There is a one-to-one map between the field-content and the interaction structure of a $D = 4$, $\mathcal{N} = 1$ gauge theory and the generalized Kronheimer algorithm of solving quotient singularities $\mathbb{C}^3/\Gamma$ via a Kähler quotient based on the McKay correspondence. All items on both sides of the one-to-one correspondence are completely determined by the structure of the finite group Γ and by its specific embedding into SU(3).

Generalized Kronheimer construction

For C^3/Γ there is a **generalized Kronheimer construction** of the resolution which is just taylored to define the building blocks of a gauge theory in D=4. This is based on a **generalized McKay correspondence.**

In D=4 we obtain an N=1 gauge theory associated with a D3-brane.

Before we inspect the Kronheimer construction in the perspective of Physics let us summarize some deep mathemtical results on the cohomolgy of the crepant resolutions of quotient singularities.

A resolution of a singularity $X \rightarrow Y$ is crepant when the canonical bundle of X is the pull-back of the canonical bundle of Y. Hence if $X = \frac{\mathbb{C}^3}{\Gamma}$ is an orbifold, its canonical bundle is trivial and such is the canonical bundle of the crepant resolution Y. Hence Y is a non-compact Calabi-Yau three-fold and admits Ricci flat metrics.

What we learn on cohomology from our friends mathematicians

Since the years 1990s to the present time there has been a quite extended activity in the mathematical community of algebraic geometers on the issue of *crepant resolutions* $Y \to \mathbf{C}^n/\Gamma$ *(n=3 in particular)* and on *the McKay correspondence*. Some theorems have been established.

Important contributions have been given by: *Y.Ito, M. Reid, A. Craw, S.S. Roan, D. Markushevich, I. Dolgachev, A. Degeratu, T. Walpuski and others*.

The main and for physicists most challenging theorem is due to Ito & Reid and it is based on the notion of **age grading** which we briefly recall in the next slide.

The age grading

Let $\Gamma \subset SU(n)$ be a finite subgroup. Hence each of its group elements has a linear action on $\mathbf{C}^n$: the $\mathcal{Q}$-representation.

$$\forall \gamma \in \Gamma : \quad \gamma.\vec{z} = \underbrace{\begin{pmatrix} \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \dots & \dots & \dots \end{pmatrix}}_{\mathcal{Q}(\gamma)} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

In a finite group every element has a finite order **p** : **γ^p =Id** (**p**=integer). Hence $\mathcal{Q}(\gamma)$ can be diagonalized and its eigenvalues are r-th roots of the unity. They will be as follows:

$$(\lambda_1, \dots, \lambda_n) = \exp\left[\frac{2\pi\, i}{p} a_i\right] \quad ; \quad p > a_i \in \mathbb{N} \quad i = 1, \dots, n$$

This introduces **age -vectors** $\mathbf{v} = \frac{1}{p}\{a_1, a_2, \dots, a_n\}$

that are clearly properties of the entire **conjugacy class C of γ**

$$\mathrm{age}\,(\gamma) = \frac{1}{p}\sum_{i=1}^{n} a_i$$ **AGE GRADING**

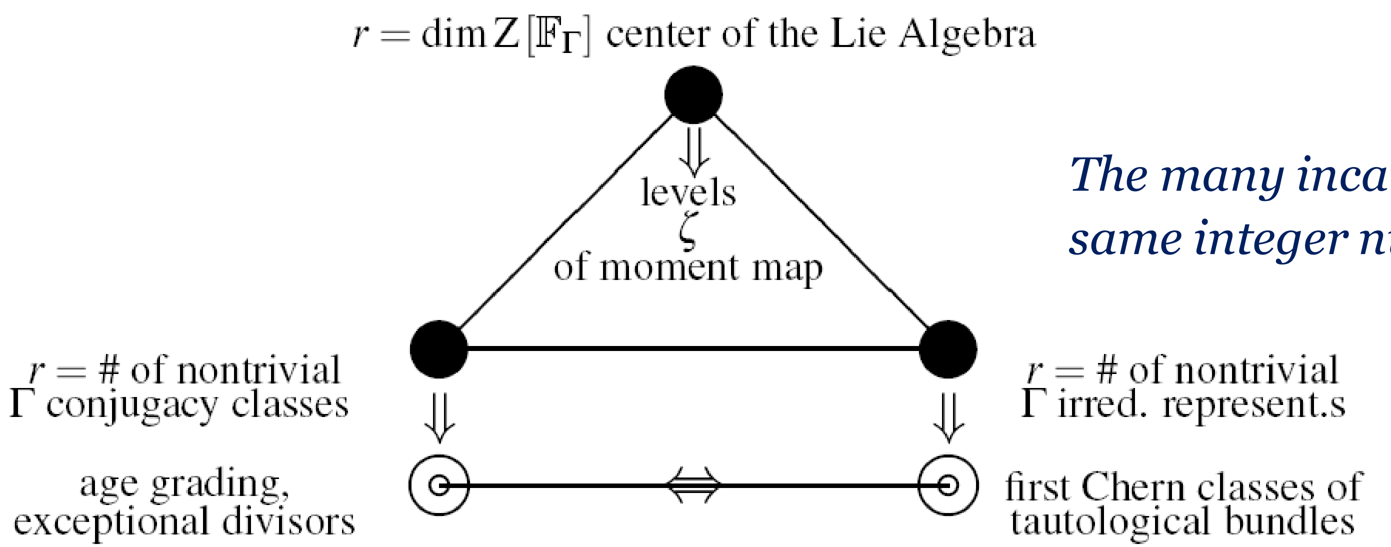
Ito Reid theorem

Theorem 4.1 *Let $Y \to \mathbb{C}^3/\Gamma$ be a crepant*[1] *resolution of a Gorenstein*[2] *singularity. Then we have the following relation between the de-Rham cohomology groups of the resolved smooth variety Y and the ages of Γ conjugacy classes:*

$$\dim H^{2k}(Y) = \#\ \textit{of age } k \textit{ conjugacy classes of } \Gamma$$

Furthermore $\dim H^{2k+1}(Y) = 0$ and the representatives of H^{2k} are actually (k,k)-forms

The age grading is not an intrinsic property of Γ, rather of its action on C^3



The many incarnations of the same integer number **r**

Terminology and some conclusions

There is a single class of age 0, namely the identity. The classes of age 1 are named junior classes. The classes of age 2 are named senior classes.

Junior classes are in one-to-one correspondence with a basis of generators of $\mathbf{H}^{(1,1)}$. These generators $\mathbf{\Omega^{(1,1)}_i}$ can be regarded as the first Chern classes of as many line-bundles $\mathcal{L}_i$ and these line bundles correspond to as many divisors $\mathcal{D}_i$. These are the components of the exceptional divisor $\mathcal{D}_E$ created by the blow-up. When an $\mathbf{\Omega^{(1,1)}_i}$ has compact support, by Poincaré duality it is dual to an $\mathbf{\Omega^{(2,2)}_i}$ belonging to $\mathbf{H}^{(2,2)}$.

These are in correspondence with the **senior classes**. **In other words the senior classes are in one-to-one correspondence with the compact components of the exceptional divisor**.

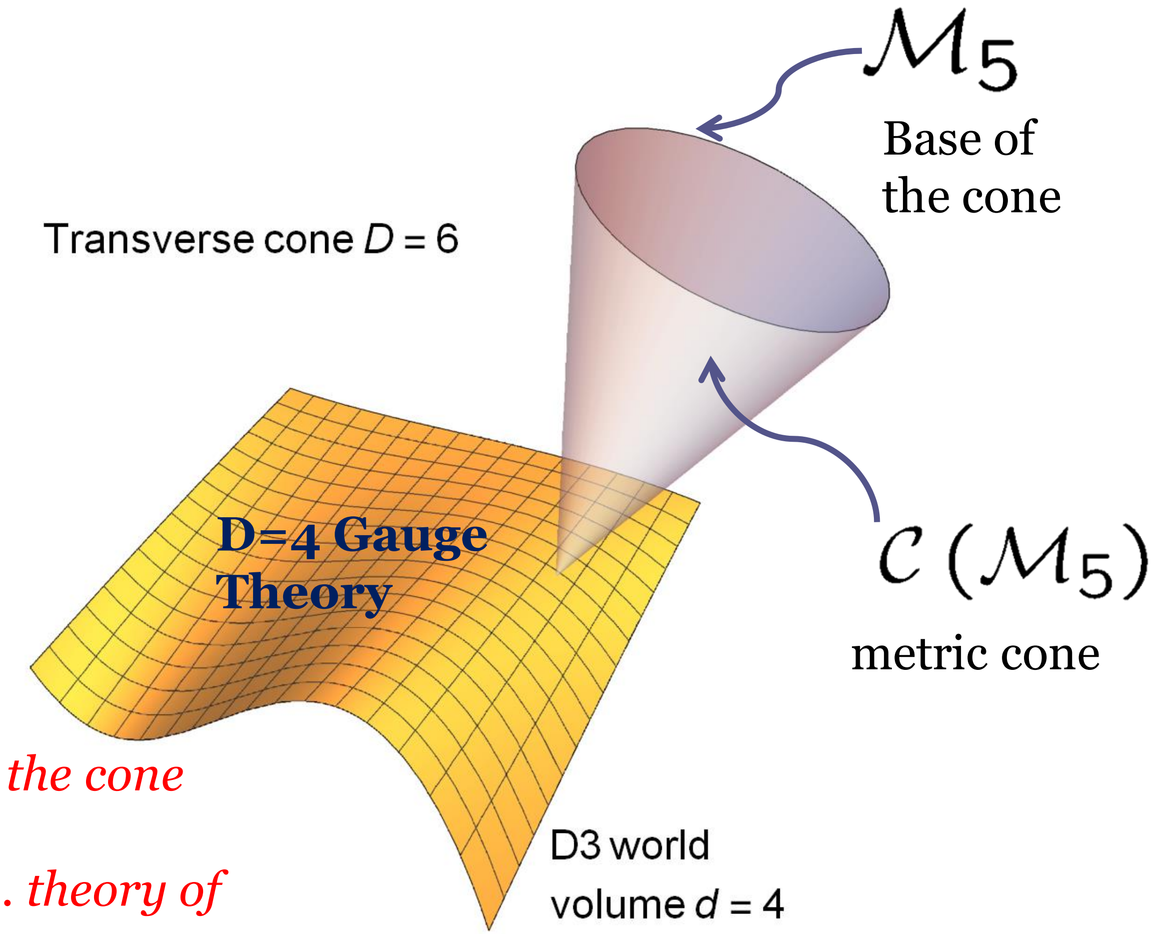
We have the 2-forms $\omega^{(1,1)}_i$ defined by **the generalized Kronheimer construction** and in one-to-one correspondence with the irreps of Γ. What is their precise relation with the $\mathbf{\Omega^{(1,1)}_i}$ and the divisors $\mathcal{D}_i$ that are in one-to-one correspondence with the conjugacy classes? ***This pairing between irreps and conjugacy classes is of the outmost interest in Physics.***

The scenario in Physics

Let us review the Physics section of our stage: Gauge/Gravity correspondence and branes

The AdS$_5$/CFT$_4$ scenario and some history: 1°

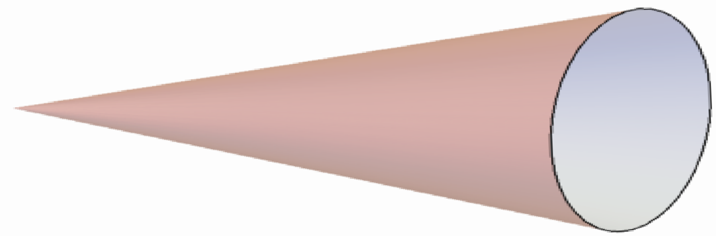
The fundamental issue is as follows.
Let us consider D3-brane solutions of D=10 type IIB SUGRA. We have a gauge theory on the D=4 world volume.
What can we learn on this gauge theory, from the geometry of the transverse cone?



Essentially everything is fixed by the cone geometry and there is a beautiful correspondence with the mathem. theory of singularity resolutions.

The AdS$_5$/CFT$_4$ scenario and some history: 2°

The classical situation studied years ago corresponds to the case where the transverse cone is the metric cone on **a Sasakian manifold M_5**



base of the fibration	projection	5-manifold	inclusion	metric cone
$\mathscr{B}_4$	$\xleftarrow{\pi}$	$\mathscr{M}_5$	$\hookrightarrow$	$\mathscr{C}(\mathscr{M}_5)$
$\Updownarrow$	$\forall p \in \mathscr{B}_4 \quad \pi^{-1}(p) \sim \mathbb{S}^1$	$\Updownarrow$		$\Updownarrow$
Kähler K_2		Sasakian		Kähler Ricci flat K_3

$$\mathrm{AdS}_5 \times \mathcal{M}_5$$

The Sasakian metric

$$ds^2_{\mathcal{M}_5} = (d\chi - \mathcal{A})^2 + g_{ij^\star}\, dz^i \otimes d\bar{z}^{j^\star}$$

D3-brane solution of D=10 IIB SUGRA

The metric in D=10

$$ds^2_{[10]} = H(\mathrm{y},\bar{\mathrm{y}})^{-\frac{1}{2}}(-\eta_{\mu\nu}\,dx^\mu \otimes dx^\nu) + H(\mathrm{y},\bar{\mathrm{y}})^{\frac{1}{2}}\left(\mathrm{g}^{\mathrm{RFK}}_{\alpha\beta^\star}\,dy^\alpha \otimes dy^{\beta^\star}\right)$$

where $$ds^2_{\mathcal{M}_6} = \mathrm{g}^{\mathrm{RFK}}_{\alpha\beta^\star}\,dy^\alpha \otimes dy^{\beta^\star}$$ Ricci flat metric on the cone M_6

The 5-form

$$F^{RR}_{[5]} = \alpha\,(U + \star_{10}U) \quad ; \quad U = d\left(H^{-1}\,\mathrm{Vol}_{\mathbb{R}(1,3)}\right)$$

The harmonic function in d=6

$$\Box_{\mathcal{M}_6} H(y) = 0$$

and define

$$H(y) \equiv 1 - \frac{1}{r(y)^4}$$

$$\begin{cases} r \to \infty = \text{asymptotic flat limit} \\ r \to 0 = \text{near horizon limit} \end{cases}$$

More precisely

Let us consider the harmonic function as a map

$$\mathfrak{H} \quad : \quad \mathcal{M}_6 \rightarrow \mathbb{R}_+$$

This introduces a foliation into a one-parameter family of 7-manifolds

$$\forall r \in \mathbb{R}_+ \quad : \quad \mathcal{M}_5(r) \equiv \mathfrak{H}^{-1}(1 - r^{-4}) \subset \mathcal{M}_6$$

In order to have the possibility of residual supersymmetries we are interested in cases where M_6 is actually a Ricci-flat Kaehler 3-fold K_3

$$K_2 \xleftarrow{\pi} \mathcal{M}_5 \quad \xleftarrow{\mathfrak{H}^{-1}} \quad K_3 \quad \overset{\mathcal{A}}{\hookrightarrow} \quad \mathbb{V}_q$$

Projection

Inclusion into a higher dimensional algebraic variety

The N=4 case with no singularity

$$\mathbb{CP}^2 \xleftarrow{\pi} \mathbb{S}^5 \overset{Cone}{\hookrightarrow} \mathbb{C}^3 \overset{\mathcal{A}=\mathrm{Id}}{\hookrightarrow} \mathbb{C}^3$$

The near horizon limit produces the standard solution of D=11 SUGRA

$$\mathrm{AdS}_5 \times \mathbb{S}^5$$

This leads to the isometry group SU(2,2|4). The Kaluza Klein states are organized into short supermultiplets of SU(2,2|4)

The singular orbifold cases

Using the Hopf fibration of the 5-sphere

$$\begin{aligned} \pi &: \mathbb{S}^5 \to \mathbb{CP}^2 \\ \forall y \in \mathbb{CP}^2 &: \pi^{-1}(y) \sim \mathbb{S}^1 \end{aligned}$$

We have

$$\frac{\mathbb{CP}^2}{\Gamma} \xleftarrow{\pi} \frac{\mathbb{S}^5}{\Gamma} \overset{Cone}{\hookrightarrow} \frac{\mathbb{C}^3}{\Gamma} \overset{\mathcal{A}=?}{\hookrightarrow} ?$$

Where Γ is a finite subgroup of SU(3) with a linear holomorphic action on $\mathbf{C}^3$

We distinguish two cases

Two cases

- A) $\Gamma \subset \mathrm{SU}(2) \subset \mathrm{SU}(2) \otimes \mathrm{U}(1) \subset \mathrm{SU}(3)$

Resolution of Kleinian
singularities à la Kronheimer

$$\frac{\mathbb{C}^3}{\Gamma} \simeq \mathbb{C} \times \left(\frac{\mathbb{C}^2}{\Gamma} \leftarrow \mathcal{M}_\zeta\right)$$

HyperKaehler quotient, N=2 susy in d=4 (McKay corr.)

- B) $\Gamma \subset \mathrm{SU}(3)$

$$\frac{\mathbb{C}^3}{\Gamma} \simeq \left(\frac{\mathbb{C}^3}{\Gamma} \leftarrow \mathcal{M}_\zeta\right)$$

Generalized Kronheimer construction
and McKay correspondence.

Kaehler quotient, N=1 susy in d=4

The map

Geometry

- S_Γ =Hom$_\Gamma$(Q× R,R) linear data, dim$_C$(S_Γ) =3 |Γ|
- G_Γ =quiver group (see later). F_Γ is the maximal compact subgroup thereof.
- The dimension is dim F_Γ = |Γ| -1
- The moment map $\mu : S_\Gamma \to \mathbf{F}_\Gamma^*$ defines |Γ| -1 functions $\mathcal{P}_I(q)$ that enter the Kaehler quotient construction
- one has to lift to level ζ_I>0 the moment maps associated with the center
- One needs a quadratic constraint $p \wedge p = 0$ that cuts a locus $V_{|\Gamma|+2}$ of dimension |Γ|+2:
- The Kaeher quotient of $V_{|\Gamma+2|}$ with respect to F_Γ is the minimal crepant resolution $M_\zeta \to C^3/\Gamma$

Gauge Theory

- S_Γ = Kaehler manifold of the Wess-Zumino multiplets (flat).
- F_Γ is the gauge group of the gauge theory
- The dimension is dim F_Γ = |Γ| -1
- The functions $\mathcal{P}_I(q)$ define the D terms and enter the formula for the scalar potential
- The level parameters ζ_I are the Fayet Iliopoulos parameters
- The equation $p \wedge p=0$ defines a universal cubic superpotential W_Γ
- The smooth manifold M_ζ is the space of vacua of the gauge theory

The final form of the scalar potential

$$V(z,\bar{z}) = \frac{1}{6}\left(\partial_i \mathcal{W}\,\partial_{j^\star}\overline{\mathcal{W}}\,g^{ij^\star} + \mathbf{m}^{\Lambda\Sigma}\left(\mathcal{P}_\Lambda - \zeta_I\,\mathfrak{C}^I_\Lambda\right)\left(\mathcal{P}_\Sigma - \zeta_J\,\mathfrak{C}^J_\Sigma\right)\right)$$

$$\mathbf{m}^{\Lambda\Sigma}(z,\bar{z}) \equiv \frac{1}{4\alpha^2}\,\kappa^{\Lambda\Gamma}\,\kappa^{\Sigma\Delta}\,k^i_\Gamma\,k^{j^\star}_\Delta\,g_{ij^\star}$$

The manifold of extrema of the scalar potential coincides with the minimal crepant resolution of the singularity $\mathbb{C}^3/\Gamma$ according with the **generalized Kronheimer construction** based on the generalized **McKay correspondence**.
Indeed since the potential is a sum of squares the extrema are defined by

$$\partial_i \mathcal{W} = 0 \quad \Longleftrightarrow \quad \mathbf{p} \wedge \mathbf{p} = 0$$

$$\mathcal{P}_\Lambda = \zeta_I\,\mathfrak{C}^I_\Lambda$$

Furthermore gauge invariance implies that we have to consider only orbits of the gauge group and this completes the Kaehler quotient procedure.

The diagram of the smooth resolution in case A)

$$\mathcal{M}_5 \xleftarrow{\mathfrak{H}^{-1}} \mathbb{C} \times ALE_\Gamma \xleftarrow{\mathrm{Id} \times qK} \mathbb{C} \times \mathbb{V}_{|\Gamma|+1} \overset{\mathcal{A}_\mathcal{P}}{\hookrightarrow} \mathbb{C} \times \mathbb{C}^{2|\Gamma|}$$

The map $\overset{\mathcal{A}_\mathcal{P}}{\hookrightarrow}$ denotes the inclusion map of the variety $\mathbb{V}_{|\Gamma|+1}$ in $\mathbb{C}^{2|\Gamma|}$

$$qK \quad : \quad \mathbb{V}_{|\Gamma|+1} \longrightarrow \mathbb{V}_{|\Gamma|+1} /\!\!/_{K} \mathcal{F}_{|\Gamma|-1} \simeq ALE_\Gamma$$

qK is the Kaehler quotient with respect to the gauge group.

Altogether we have a HyperKaehler quotient

$$ALE_\Gamma = \mathbb{C}^{2|\Gamma|} /\!\!/_{HK} \mathcal{F}_{|\Gamma|-1}$$

It is convenient to split the HK quotient into two steps in order to compare with the case C^3/Γ

The diagram of the smooth resolution in case B)

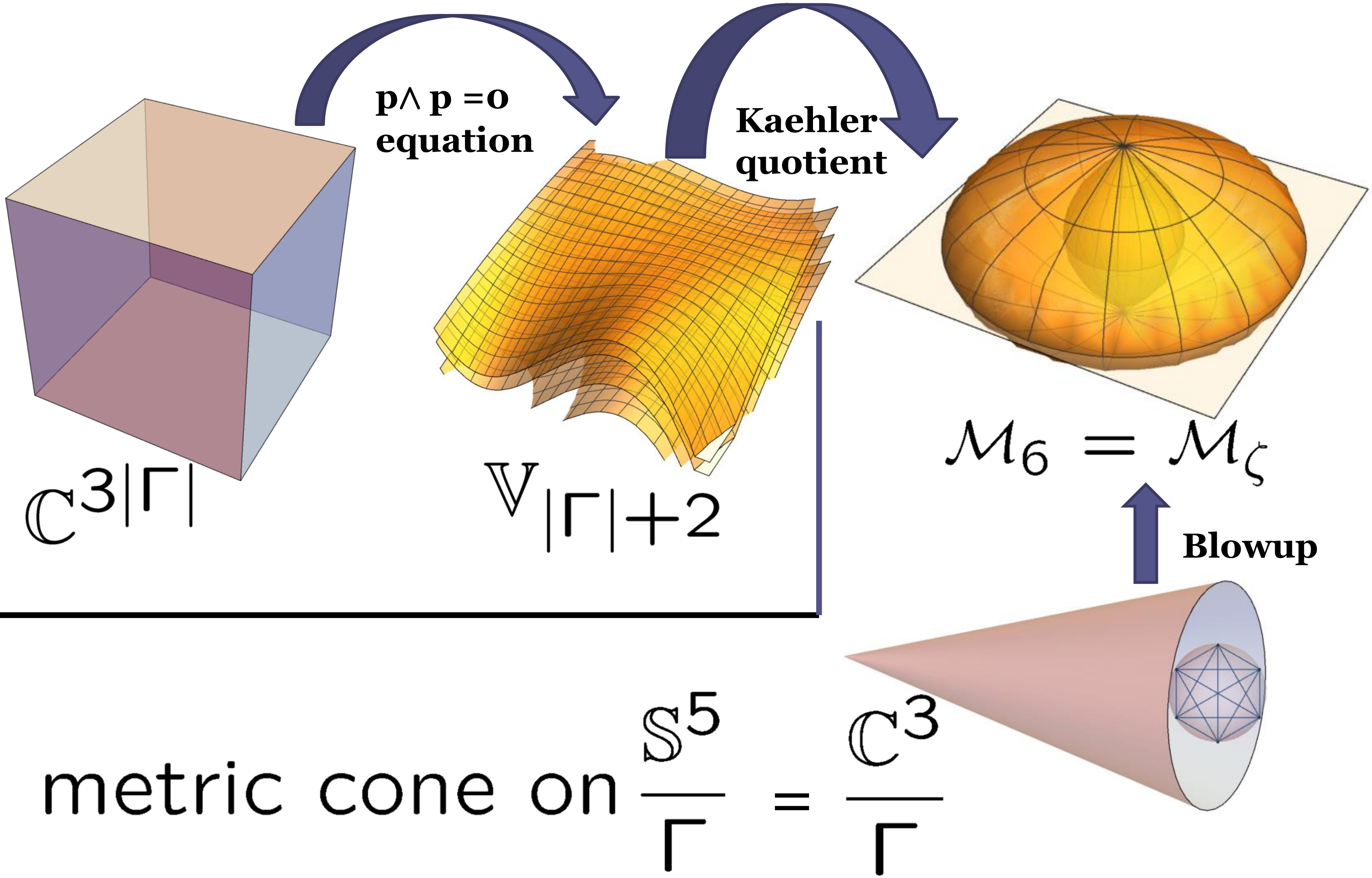
$$\mathcal{M}_5 \xleftarrow{\mathfrak{H}^{-1}} Y_\Gamma \xleftarrow{qK} \mathbb{V}_{|\Gamma|+2} \stackrel{\mathcal{A}_\mathcal{P}}{\hookrightarrow} \mathbb{C}^{3|\Gamma|}$$

The intermediate step, just as in case A) is a Kaehler quotient, yet the starting point variety $V_{|\Gamma|+2}$ has a different definition. From the physical viewpoint we have N=1 rather than N=2 susy and the superpotential is not defined by the holomorphic moment maps, mathematically this corresponds to the fixed equation $p \wedge p = 0$ that amounts to identifying $V_{|\Gamma|+2}$ with a certain orbit with respect to the quiver group G_Γ, actually the compactification of the gauge group F_Γ:

$$\mathbb{V}_{|\Gamma|+2} = \mathcal{D}_\Gamma \equiv \mathrm{Orbit}_{\mathcal{G}_\Gamma}(L_\Gamma)$$

We see later what the locus L_Γ is.

A visual scheme



The McKay correspondence for $\frac{\mathbb{C}^2}{\Gamma}$

In finite group theory we have the decomposition of any rep. D into irreps D_μ

$$D = \bigoplus_{\mu=1}^{r} a_\mu D_\mu$$

$$a_\mu = \frac{1}{g} \sum_{i=1}^{r} g_i \chi_i^{(D)} \chi_i^{(\mu)\star}$$

Let Q be the defining rep. of $\Gamma \subset SU(2)$

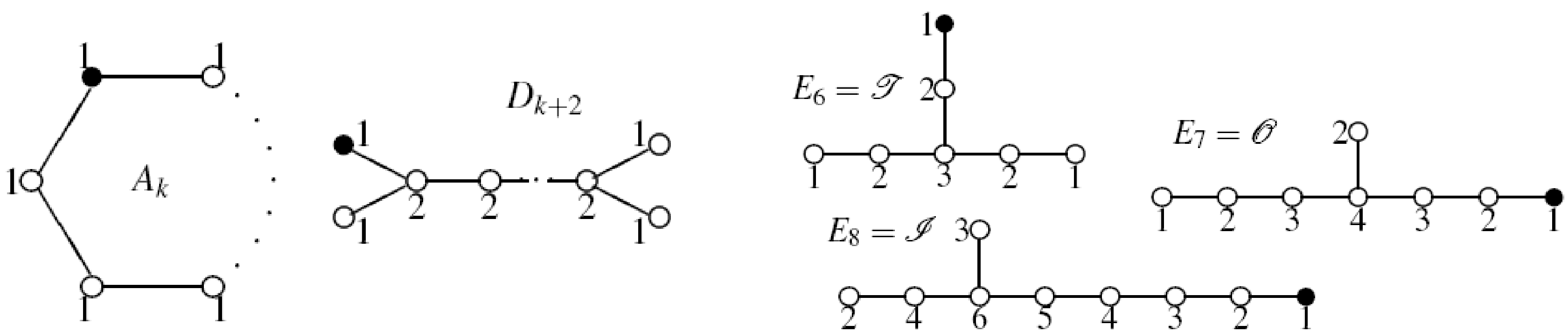
$$Q \otimes D_\mu = \bigoplus_{\nu=0}^{r} A_{\mu\nu} D_\nu$$

The isomorphic ADE classification of Kleinian groups Γ and semisimple Lie algebras is known. The Coxeter numbers coincide with the dimensions of Γ irreps

Miracolous properties the matrix

$$\mathbf{c_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}}$$

It is the extended Cartan matrix of the Lie algebras **A_k, D_{k+2}, E_6, E_7, E_8**



McKay correspondence and the Kronheimer construction of ALE manifolds

Define a space $\mathcal{P}$ of pairs p= (A,B) of complex $|\Gamma| \times |\Gamma|$ matrices. Define the action of the group Γ on $\mathcal{P}$

$$\forall \gamma \in \Gamma \quad : \quad \begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \mathcal{Q}_\gamma \begin{pmatrix} R(\gamma)\, A\, R(\gamma^{-1}) \\ R(\gamma)\, B\, R(\gamma^{-1}) \end{pmatrix}$$

where R(γ) is the regular representation and $\mathcal{Q}_\gamma$ the defining representation. In intrinsic notation

$$\mathcal{P} \simeq \mathrm{Hom}\,(R, \mathcal{Q} \otimes R)$$

Introduce the Γ invariant subspace of $\mathcal{P}$

$$\mathcal{S} \equiv \{p \in \mathcal{P} / \forall \gamma \in \Gamma\,,\, \gamma \cdot p = p\} = \mathrm{Hom}_\Gamma\,(R, \mathcal{Q} \otimes R)$$

The space $\mathcal{S}$ is a flat Kaehler manifold with **complex dimension 2|Γ|** which encodes (in Physics) the **Wess-Zumino multiplets** of the CS theory (actually) **hypermultiplets** since susy will be N=2.

Why the dimension of $\mathcal{S}$ is $2|\Gamma|$?

The answer is:

1. **McKay correspondence**
2. **Regular representation**
3. **Schur's Lemma**

$$\mathcal{S} = \bigoplus_{\mu,\nu} A_{\mu,\nu} \mathrm{Hom}(\mathbb{C}^{n_\mu}, \mathbb{C}^{n_\nu})$$

$$\dim_{\mathbb{C}} \left[\mathrm{Hom}_{\Gamma}(R, \mathcal{Q} \otimes R)\right] = 2|\Gamma|$$

Actually the space $\mathcal{S}$ is a flat HyperKaehler manifold with a triplet of Kaehler forms arranged into a quaternion

$$\Theta = \mathrm{Tr}(\mathrm{d}p^\dagger \wedge \mathrm{d}p) = \begin{pmatrix} \mathrm{i}\,\mathbf{K} & \mathrm{i}\overline{\mathbf{\Omega}} \\ \mathrm{i}\,\mathbf{\Omega} & -\mathrm{i}\,\mathbf{K} \end{pmatrix}$$

$$\begin{aligned}
\mathbf{K} &= -\mathrm{i}\left[\mathrm{Tr}\left(\mathrm{d}A^\dagger \wedge \mathrm{d}A\right) + \mathrm{Tr}\left(\mathrm{d}B^\dagger \wedge \mathrm{d}B\right)\right] \equiv \mathrm{i} g_{\alpha\bar{\beta}}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^{\bar{\beta}} \\
ds^2 &= g_{\alpha\bar{\beta}}\, \mathrm{d}q^\alpha \otimes \mathrm{d}q^{\bar{\beta}} \\
\mathbf{\Omega} &= 2\mathrm{Tr}\left(\mathrm{d}A \wedge \mathrm{d}B\right) \equiv \Omega_{\alpha\beta}\, \mathrm{d}q^\alpha \wedge \mathrm{d}q^\beta
\end{aligned}$$

This allows to perform a HyperKaehler quotient with respect to a ***suitable gauge group*** $\mathcal{F}_\Gamma$

The *Discreet Charm* of the integer ***r***

The integer ***r*** counts several distinct things at the same time

1. The number of non trivial **irreps** of Γ.
2. The number of non trivial **conjugacy classes** of Γ.
3. The dimension of the center $\mathfrak{z}[\mathbb{F}_\Gamma]$ of the gauge Lie algebra.
4. Hence the number of **Fayet Iliopoulos parameters** in the CS supergauge theory.
5. As we will see also the number of **tautological holomorphic bundles** on the resolved variety: $M_\zeta \rightarrow \mathbf{C}^n/\Gamma$ (n=2,3)
6. In the case n=2 (ADE) the **rank** of the semisimple Lie algebra corresponding to Γ.

The resolved smooth manifold ALE_Γ is obtained as the **HyperKaeler quotient** of $\mathcal{S}_\Gamma$ by $\mathcal{F}_\Gamma$

$$\mathcal{M}_\zeta \equiv \mu^{-1}(\vec{\zeta}) \,/\!/\, \mathcal{F}_\Gamma$$

where $$\vec{\zeta} \in \mathbb{R}^3 \otimes \mathfrak{z}\left[\mathbb{F}_\Gamma\right]^\star$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 1°

Next let $\Gamma \subset$ **SU(n).** We have a generalized McKay correspondence

$$\mathcal{Q} \otimes \mathsf{D}_i = \bigoplus_{j=1}^{r+1} \mathcal{A}_{ij}\, \mathsf{D}_j$$

$$\bar{c}_{ij} = n\,\delta_{ij} - \mathcal{A}_{ij}$$

generalized extended Cartan matrix

$$\mathbf{n} \equiv \{1, n_1, \ldots, n_r\}$$

vector of irrep dimensions

$$\bar{c}.\mathbf{n} = 0$$

fundamental property

For n=3 we introduce a space $\mathcal{P}_\Gamma$ of triplets of $|\Gamma| \times |\Gamma|$ matrices

$$p \in \mathcal{P}_\Gamma \equiv \mathsf{Hom}\,(R, \mathcal{Q} \otimes R) \Rightarrow p = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 2°

Similarly we define the invariant subspace

$$\mathcal{S}_\Gamma \equiv \mathrm{Hom}_\Gamma(R, Q \otimes R) = \{p \in \mathcal{P}_\Gamma / \forall \gamma \in \Gamma, \gamma \cdot p = p\}$$

where the group action is

$$\forall \gamma \in \Gamma : \quad \gamma \cdot p \equiv \mathcal{Q}(\gamma) \begin{pmatrix} R(\gamma)\, A\, R(\gamma^{-1}) \\ R(\gamma)\, B\, R(\gamma^{-1}) \\ R(\gamma)\, C\, R(\gamma^{-1}) \end{pmatrix}$$

Because of the McKay relation we have

$$\mathcal{S}_\Gamma = \bigoplus_{i,j} A_{i,j} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

⬇

$$\dim_{\mathbb{C}} \mathcal{S}_\Gamma = 3 \sum_i n_i^2 = 3|\Gamma|$$

The space $\mathcal{S}_\Gamma$ is a flat Kaehler manifold of dimension $3|\Gamma|$. It accomodates the WZ multiplets of the N=1 D=4 gauge theory. So there are no holomorphic moment maps but we can have a superpotential $\mathcal{W}$ the vanishing of whose derivatives provides holomorphic constraints.

The *gauge group* and the *quiver group*

$$\mathcal{F}_\Gamma = \bigotimes_{\mu=0}^{r} \mathsf{U}(\mathsf{n}_\mu) \bigcap \mathsf{SU}(|\Gamma|)$$

gauge group

$$\mathcal{G}_\Gamma = \bigotimes_{\mu=0}^{r} \mathsf{GL}(\mathsf{n}_\mu, \mathbb{C}) \bigcap \mathsf{SL}(|\Gamma|, \mathbb{C})$$

quiver group

The gauge group is the maximal compact subgroup of the quiver group, the latter being the complexification of the former. The real dimension of the gauge group is |Γ|-1, the complex dimension of the quiver group is the same. We have the ***real moment map***, well known in supersymmetric gauge theories (D-terms)

$$\mu \ : \ \mathcal{S}_\Gamma \longrightarrow \mathbb{F}^\star_\Gamma$$

dual of the gauge Lie algebra

real moment maps

$$\mu(p) = -i\left([A, A^\dagger] + [B, B^\dagger]\right)$$

$$\mathfrak{P}_A = \mathsf{Tr}\left(\mu(p)\, f_A\right)$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 3°

How can we step down from **$3|\Gamma|$** complex dimensions to **3-dimensions**?
The gauge group $\mathcal{F}_\Gamma$ has **$|\Gamma|$-1 generators** and the corresponding Kaehler quotient kills $|\Gamma|$-1 complex parameters. Hence the starting point should be a **variety with complex dimensions $|\Gamma|$ +2.**

Question: *what is the analogue of holomorphic moment map equation?*

Answer: it is $\mathbf{p} \wedge \mathbf{p} = 0$

$$0 = \epsilon^{ijk} \mathbf{p}_i \cdot \mathbf{p}_j$$
$$\Updownarrow$$
$$0 = [A, B] = [B, C] = [C, A]$$

The general solution to this constraint is given by a variety $\mathbf{V}_{|\Gamma|+2}$ that can be seen as the quiver group orbit of a special 3-dimensional locus

$$\mathbb{V}_{|\Gamma|+2} = \mathcal{D}_\Gamma \equiv \text{Orbit}_{\mathcal{G}_\Gamma}(L_\Gamma)$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 4°

$$\mathcal{S}_\Gamma \supset L_\Gamma \equiv \left\{ \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \in \mathcal{S}_\Gamma \mid A_0, B_0, C_0 \text{ diagonal in natural basis of R} \right\}$$

The locus L_Γ is easily seen to be 3-dimensional and we have

$$\mu^{-1}(0) = \mathrm{Orbit}_{\mathcal{F}_\Gamma}(L_\Gamma)$$

Hence L_Γ describes the singular orbifold $\mathbb{C}^3/\Gamma$

Introducing the orthogonal decomposition

$$\mathbb{G}_\Gamma = \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma$$

$$[\mathbb{F}_\Gamma, \mathbb{F}_\Gamma] \subset \mathbb{F}_\Gamma \quad ; \quad [\mathbb{F}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{K}_\Gamma \quad ; \quad [\mathbb{K}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{F}_\Gamma$$

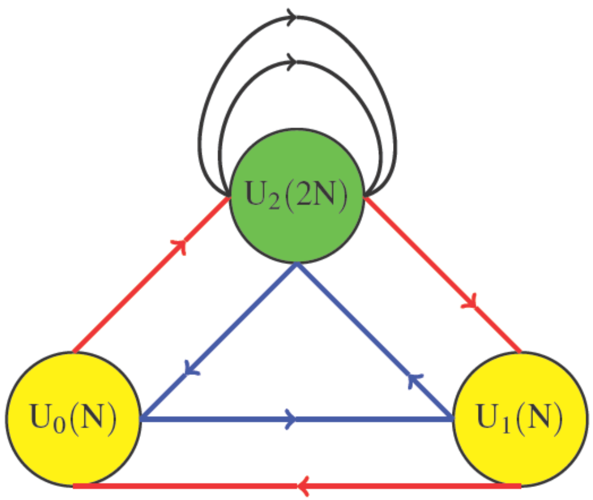
$$\mathcal{D}_\Gamma = \mathrm{Orbit}_{\mathcal{F}_\Gamma}(\exp[\mathbb{K}_\Gamma] \cdot L_\Gamma)$$

$$\mu^{-1}(\zeta) /\!/ \mathcal{F}_\Gamma = \left\{ Z \in \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \parallel \begin{array}{ll} \mu_I(Z) = 0 & \text{if } f_I \notin \mathfrak{Z} \\ \mu_I(Z) = \zeta_I & \text{if } f_I \in \mathfrak{Z} \end{array} \right\}$$

Dih_3

$$A^3 = 1 \quad ; \quad B^2 = 1 \quad ; \quad (AB)^2 = \mathbf{1}$$

$$A = \begin{pmatrix} e^{\frac{2\pi}{3}i} & 0 & 0 \\ 0 & e^{-\frac{2\pi}{3}i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



Field	$U_0(1)\times U_1(1)\times U_2(2)$	#of components
$\Phi_{0,1}$	$(1,\bar{1},1)$	1
$\Phi_{1,0}$	$(\bar{1},1,1)$	1
$\Phi_{0,2}$	$(\bar{1},1,2)$	2
$\Phi_{2,1}$	$(1,1,\bar{2})$	2
$\Phi_{1,2}$	$(1,\bar{1},2)$	2
$\Phi_{2,1}$	$(1,1,\bar{2})$	2
$\Phi_{2,2}$	$(1,1,4)$	4
$\Phi'_{2,2}$	$(1,1,4)$	4
		18

1. age = 0 ; $1\{0,0,0\}$
2. age = 1 ; $\frac{1}{2}\{1,1,0\}$
3. age = 1 ; $\frac{1}{3}\{0,2,1\}$

$$h^{0,0} = 1,\ h^{1,1} = 2,\ h^{2,2} = 0,$$

PSL(2,7) The second smallest simple group |PSL(2,7)|=168

$$\mathrm{PSL}(2,7) = \left(R, S, T \parallel R^2 = S^3 = T^7 = RST = (TSR)^4 = \mathbf{e}\right)$$

$$\mathbf{Irreps} = \{8, 7, 6, 3, \bar{3}\}$$

Conjugacy class of PSL(2,7)	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$
representative of the class	$\mathbf{e}$	R	S	TSR	T	SR
order of the elements in the class	1	2	3	4	7	7
age	0	1	1	1	1	2
number of elements in the class	1	21	56	42	24	24

$$h^{(1,1)}\left(\mathscr{M}_{\zeta|\mathrm{PSL}(2,7)}\right) = 4 \quad ; \quad h^{(2,2)}\left(\mathscr{M}_{\zeta|\mathrm{PSL}(2,7)}\right) = 1$$

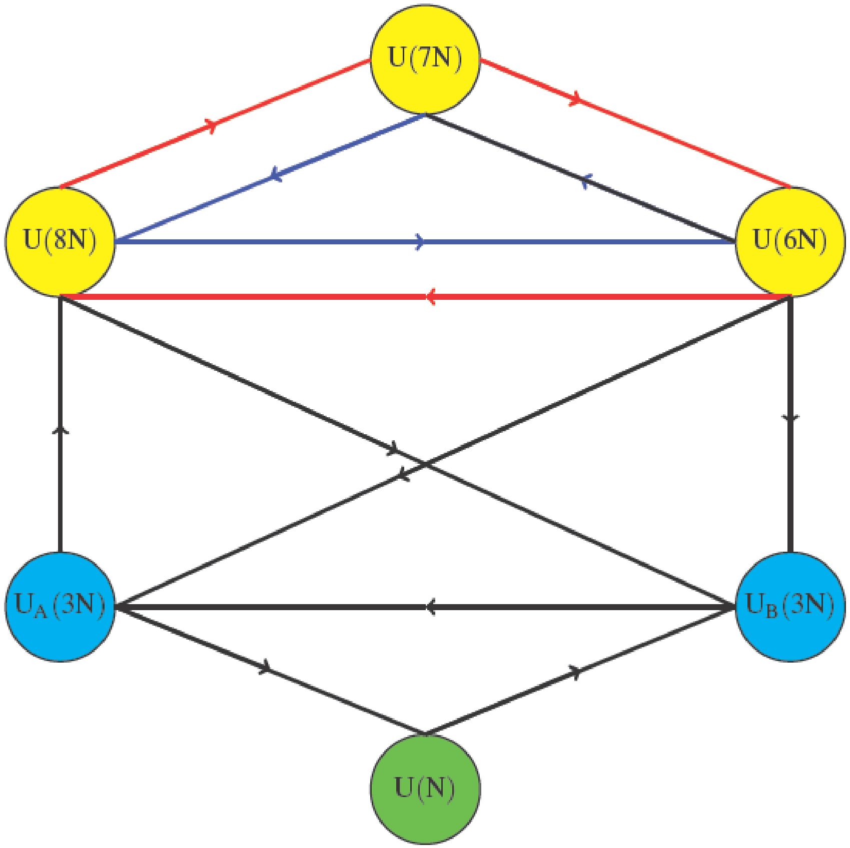


Figure 6.2: The quiver diagram of the finite group PSL(2,7) ⊂ SU(3)

This complicated Gauge Theory has not yet been constructed explicitly but the resolved manifold was obtained by Markushevitch already in 1997 using blowups of Algebraic Geometry without the use of the Kronheimer construction unknown to him at the time.

The exceptional compact divisor is described by Markushevich but not yet appropriately described as a toric variety and the Ricci flat metric on its line bundle is unknown

$$\mathcal{M}_{\zeta|\mathrm{PSL}(2,7)} = \mathcal{L}\left(\mathcal{M}_{B|\mathrm{PSL}(2,7)}\right) \xrightarrow{\pi} \mathcal{M}_{B|\mathrm{PSL}(2,7)}$$

$$\forall p \in \mathcal{M}_{B|\mathrm{PSL}(2,7)} \quad \pi^{-1}(p) \sim \mathbb{C}$$

The moment map equation

The solution of the singularity resolution problem is finally reduced to an algebraic equation for the coset element

$$\mathcal{V} = \exp\left[\mathbf{\Phi}\right] \quad ; \quad \mathbf{\Phi} \in \mathbb{K}_{\Gamma}$$

$$\mathcal{V} = \left(\begin{array}{|c|c|c|c|c|}
\hline
\mathfrak{H}_0 & 0 & 0 & \dots & 0 \\
\hline
0 & \mathfrak{H}_1 \otimes \mathbf{1}_{n_1 \times n_1} & 0 & \dots & \vdots \\
\hline
0 & 0 & \mathfrak{H}_2 \otimes \mathbf{1}_{n_2 \times n_2} & \dots & \vdots \\
\hline
\vdots & \dots & \dots & \dots\dots & \vdots \\
\hline
\vdots & \dots & \dots & \dots & 0 \\
\hline
0 & \dots & \dots & 0 & \mathfrak{H}_r \otimes \mathbf{1}_{n_r \times n_r} \\
\hline
\end{array}\right)$$

Such that

$$\mu\left(\mathcal{V} \cdot L_{\Gamma}\right) = \zeta$$

Typically that above is a system of algebraic equations of higher order. In few cases one can reduce it to order 4°, 3° or 2° obtaining solutions by radicals.

The tautological bundles

From the coset element $\mathcal{U}$ we extract a hermitian matrix

$$\mathcal{H} \equiv \begin{pmatrix} \mathfrak{H}_1 & 0 & \dots & \dots & 0 \\ 0 & \mathfrak{H}_2 & \dots & \dots & \vdots \\ \vdots & \dots\dots & \dots\dots & \dots & \vdots \\ \vdots & \dots\dots & \dots & \mathfrak{H}_{r-1} & 0 \\ 0 & \dots\dots & \dots & 0 & \mathfrak{H}_r \end{pmatrix}$$

that is the fiber metric on the direct sum

$$\mathcal{R} = \bigoplus_{i=1}^{r} \mathcal{R}_i$$

of **r** tautological bundles that, by construction, are holomorphic vector bundles with **rank** equal to the dimensions $\mathbf{n_i}$ of the **r** irreps of Γ:

$$\mathcal{R}_i \xrightarrow{\pi} \mathcal{M}_\zeta \quad ; \quad \forall p \in \mathcal{M}_\zeta \quad : \quad \pi^{-1}(p) \approx \mathbb{C}^{n_i}$$

Provided we are able to solve the moment map equation we can evaluate the first Chern classes of these bundles

$$\omega_i^{(1,1)} \equiv c_1(\mathcal{R}_i) = \frac{i}{2\pi} \overline{\partial} \partial \log \left[\mathrm{Det} \left(\mathfrak{H}_i \right) \right]$$

One simple master example

$$\frac{\mathbb{C}^3}{\mathbb{Z}_3}$$ generated by $$Y = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix}$$ $\xi^3 = 1$

$$a - \text{vectors} = \left\{\{0,0,0\}, \frac{1}{3}\{1,1,1\}, \frac{1}{3}\{2,2,2\}\right\}$$

Hence we conclude that the Hodge numbers of the resolved variety should be $h^{(0,0)} = 1$; $h^{(1,1)} = 1$; $h^{(2,2)} = 1$.

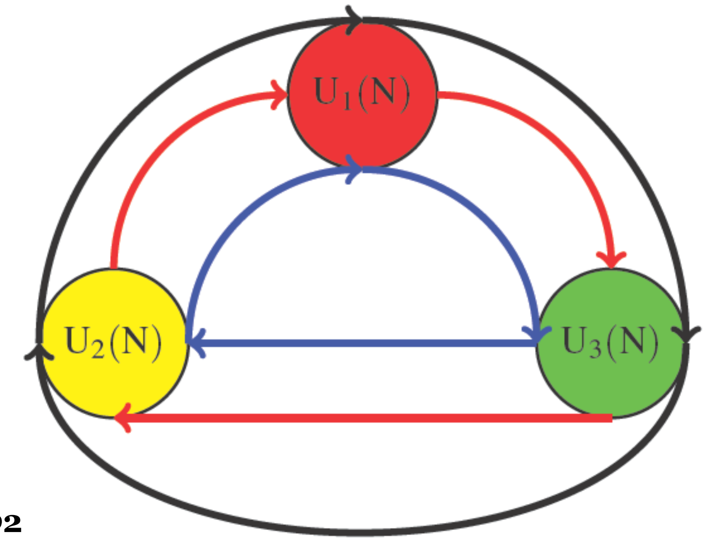
Following the generalized Kronheimer construction one arrives at the following system of algebraic equations for the entries of the H-matrix (moment map equation)

$$\left\{\frac{\Sigma\left(-1+\Upsilon_1^3\right)}{\Upsilon_1\Upsilon_2}, \frac{\Sigma\left(-1+\Upsilon_2^3\right)}{\Upsilon_1\Upsilon_2}\right\} = \left\{\zeta_1, \zeta_2\right\}$$

where

$$\Sigma = \sum_{i=1}^{3} |z_i|^2$$

Z_3 diagonal



The abstract definition of the resolved variety

$$Y_{[3]}^{\mathbb{Z}_3} = \mathscr{O}_{\mathbb{P}^2}(-3) \longrightarrow \frac{\mathbb{C}^3}{\mathbb{Z}_3}$$

A line bundle on the exceptional compact divisor **P^2**

$$\Sigma = \left(1 + |u|^2 + |v|^2\right) |w|^{2/3}$$

$$\mathscr{K}_{Rflat}(\Sigma) = \frac{2\left(\Sigma^3+1\right) - \left(\frac{1}{\Sigma^3}+1\right)^{2/3} {}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{5}{3}; -\frac{1}{\Sigma^3}\right)}{2\left(\Sigma^3+1\right)^{2/3}}$$

$$\mathcal{K}_{Rflat} \overset{w \to 0}{\longrightarrow} \log[1 + |u|^2 + |v|^2] + \ldots$$

Fubini Study metric on Excep. Divisor $\mathbb{P}^2$

Thanks to Cardano & Tartaglia!

The moment map equation is solvable by radicals!
We can explicitly calculate the $\omega^{(1,1)}{}_{1,2}$ forms

$$\omega_{1,2}^{(1,1)} = \frac{i}{2\pi}\left(\frac{d}{\mathrm{d}\Sigma}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] d\bar{z}^i \wedge \mathrm{d}z^i + \frac{d^2}{\mathrm{d}\Sigma^2}\mathrm{Log}\left[\Upsilon_{1,2}(\Sigma)\right] z^j \bar{z}^i \mathrm{d}z^i \wedge d\bar{z}^j\right)$$

Introducing the intersection integral

$$\mathcal{I}_{abc} = \int_{\mathcal{M}} \omega_a^{(1,1)} \wedge \omega_b^{(1,1)} \wedge \omega_b^{(1,1)}$$

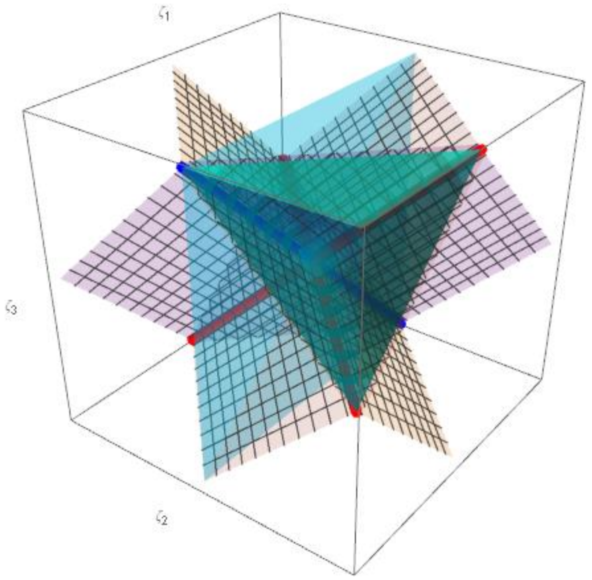
We find

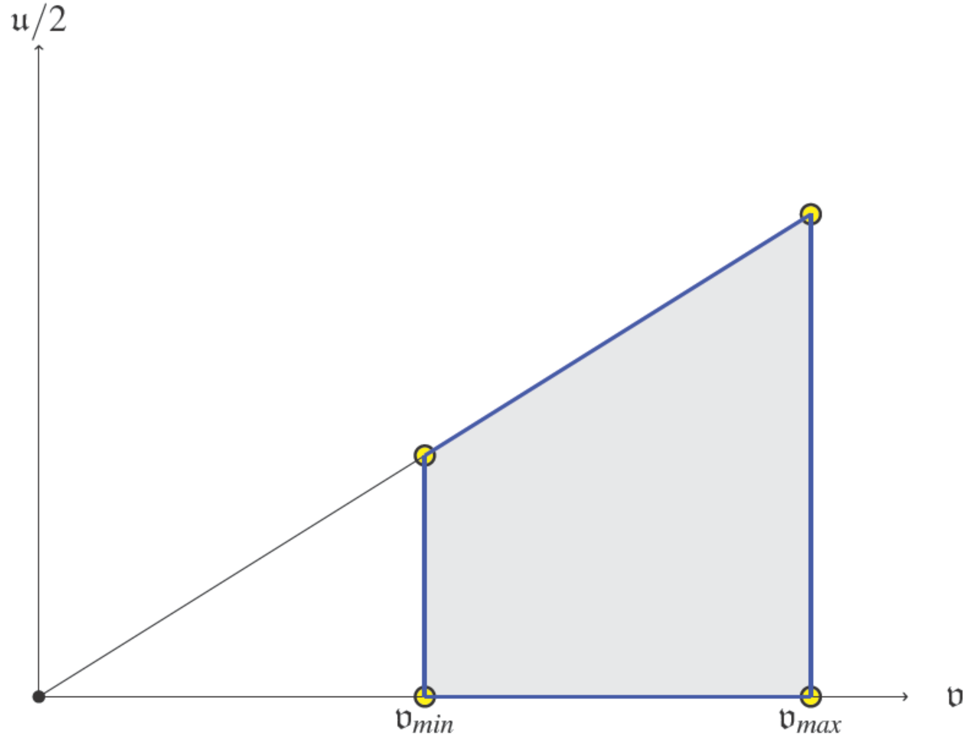
$$(\zeta_1 > 0, \zeta_2 = 0) \quad : \quad \mathscr{I}_{111} = \tfrac{1}{8}$$
$$(\zeta_1 = 0, \zeta_2 \geq 0) \quad : \quad \mathscr{I}_{111} = 0$$
$$(\zeta_1 > 0, \zeta_2 > 0) \quad : \quad \mathscr{I}_{111} = 1$$

$$(\zeta_1 > 0, \zeta_2 = 0) \quad : \quad \mathscr{I}_{211} = 0$$
$$(\zeta_1 = 0, \zeta_2 \geq 0) \quad : \quad \mathscr{I}_{211} = 0$$
$$(\zeta_1 > 0, \zeta_2 > 0) \quad : \quad \mathscr{I}_{211} = 1$$

SECOND LECTURE

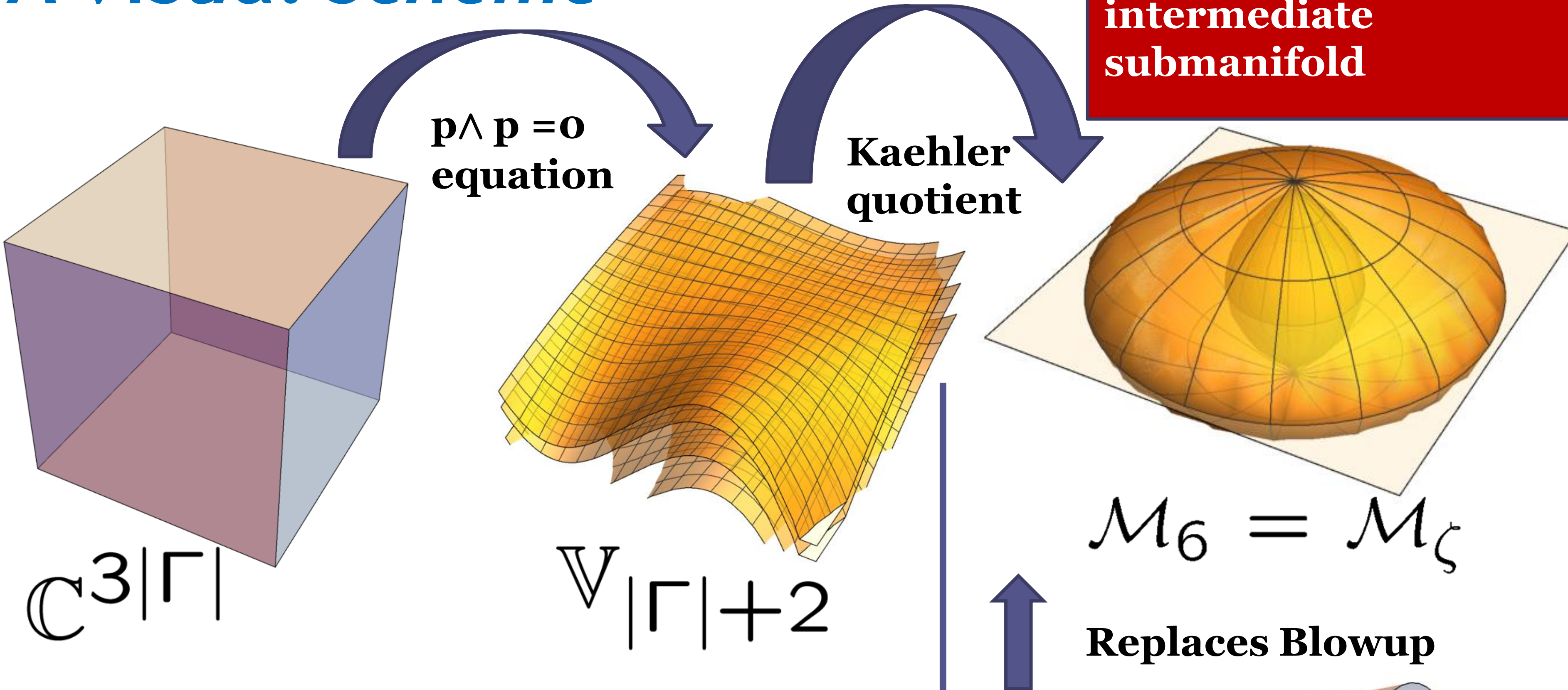
The AMSY symplectic formalism for Kaehler Geometry, the issue of Ricci flat metrics polytopes and all that….

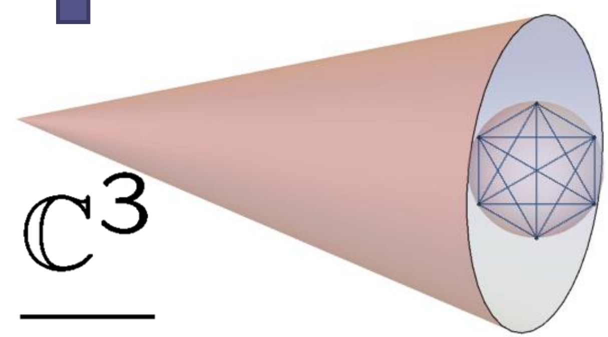




A visual scheme

Typically no 5-dim Sasaki Einstein intermediate submanifold





$$\text{metric cone on } \frac{\mathbb{S}^5}{\Gamma} = \frac{\mathbb{C}^3}{\Gamma}$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 1°

Next let $\Gamma \subset$ **SU(n).** We have a generalized McKay correspondence

$$\mathcal{Q} \otimes \mathsf{D}_i = \bigoplus_{j=1}^{r+1} \mathcal{A}_{ij}\, \mathsf{D}_j$$

$$\bar{c}_{ij} = n\,\delta_{ij} - \mathcal{A}_{ij}$$

generalized extended Cartan matrix

$$\mathbf{n} \equiv \{1, n_1, \ldots, n_r\}$$

vector of irrep dimensions

$$\bar{c}.\mathbf{n} = 0$$

fundamental property

For n=3 we introduce a space $\mathcal{P}_\Gamma$ of triplets of $|\Gamma| \times |\Gamma|$ matrices

$$p \in \mathcal{P}_\Gamma \equiv \mathsf{Hom}\,(R, \mathcal{Q} \otimes R) \Rightarrow p = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 2°

Similarly we define the invariant subspace

$$\mathcal{S}_\Gamma \equiv \mathrm{Hom}_\Gamma(R, Q \otimes R) = \{p \in \mathcal{P}_\Gamma / \forall \gamma \in \Gamma, \gamma \cdot p = p\}$$

where the group action is

$$\forall \gamma \in \Gamma : \quad \gamma \cdot p \equiv \mathcal{Q}(\gamma) \begin{pmatrix} R(\gamma)\, A\, R(\gamma^{-1}) \\ R(\gamma)\, B\, R(\gamma^{-1}) \\ R(\gamma)\, C\, R(\gamma^{-1}) \end{pmatrix}$$

Because of the McKay relation we have

$$\mathcal{S}_\Gamma = \bigoplus_{i,j} A_{i,j} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$$

⇓

$$\dim_{\mathbb{C}} \mathcal{S}_\Gamma = 3 \sum_i n_i^2 = 3|\Gamma|$$

The space $\mathcal{S}_\Gamma$ is a flat Kaehler manifold of dimension $3|\Gamma|$. It accomodates the WZ multiplets of the N=1 D=4 gauge theory. So there are no holomorphic moment maps but we can have a superpotential $\mathcal{W}$ the vanishing of whose derivatives provides holomorphic constraints.

The *gauge group* and the *quiver group*

$$\mathcal{F}_\Gamma = \bigotimes_{\mu=0}^{r} \mathsf{U}(\mathsf{n}_\mu) \bigcap \mathsf{SU}(|\Gamma|)$$

gauge group

$$\mathcal{G}_\Gamma = \bigotimes_{\mu=0}^{r} \mathsf{GL}(\mathsf{n}_\mu, \mathbb{C}) \bigcap \mathsf{SL}(|\Gamma|, \mathbb{C})$$

quiver group

The gauge group is the maximal compact subgroup of the quiver group, the latter being the complexification of the former. The real dimension of the gauge group is |Γ|-1, the complex dimension of the quiver group is the same. We have the ***real moment map***, well known in supersymmetric gauge theories (D-terms)

$$\mu \ : \ \mathcal{S}_\Gamma \longrightarrow \mathbb{F}^\star_\Gamma$$

dual of the gauge Lie algebra

real moment maps

$$\mu(p) \ = \ -i\left([A, A^\dagger] + [B, B^\dagger]\right)$$

$$\mathfrak{P}_A \ = \ \mathsf{Tr}\left(\mu(p)\, f_A\right)$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 3°

How can we step down from **$3|\Gamma|$** complex dimensions to **3-dimensions**?
The gauge group $\mathcal{F}_\Gamma$ has **$|\Gamma|$-1 generators** and the corresponding Kaehler quotient kills $|\Gamma|$-1 complex parameters. Hence the starting point should be a **variety with complex dimensions $|\Gamma|$ +2.**

Question: *what is the analogue of holomorphic moment map equation?*

Answer: it is $\mathbf{p} \wedge \mathbf{p} = 0$

$$0 = \epsilon^{ijk} \mathbf{p}_i \cdot \mathbf{p}_j$$
$$\Updownarrow$$
$$0 = [A, B] = [B, C] = [C, A]$$

The general solution to this constraint is given by a variety $\mathbf{V}_{|\Gamma|+2}$ that can be seen as the quiver group orbit of a special 3-dimensional locus

$$\mathbb{V}_{|\Gamma|+2} = \mathcal{D}_\Gamma \equiv \text{Orbit}_{\mathcal{G}_\Gamma}(L_\Gamma)$$

n=3 generalization of the McKay corr. and Kronheimer construction STEP 4°

$$\mathcal{S}_\Gamma \supset L_\Gamma \equiv \left\{ \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \in \mathcal{S}_\Gamma \mid A_0, B_0, C_0 \ \text{diagonal in natural basis of R} \right\}$$

The locus L_Γ is easily seen to be 3-dimensional and we have

$$\mu^{-1}(0) = \text{Orbit}_{\mathcal{F}_\Gamma}(L_\Gamma)$$

Hence L_Γ describes the singular orbifold C^3/Γ

Introducing the orthogonal decomposition

$$\mathbb{G}_\Gamma = \mathbb{F}_\Gamma \oplus \mathbb{K}_\Gamma$$

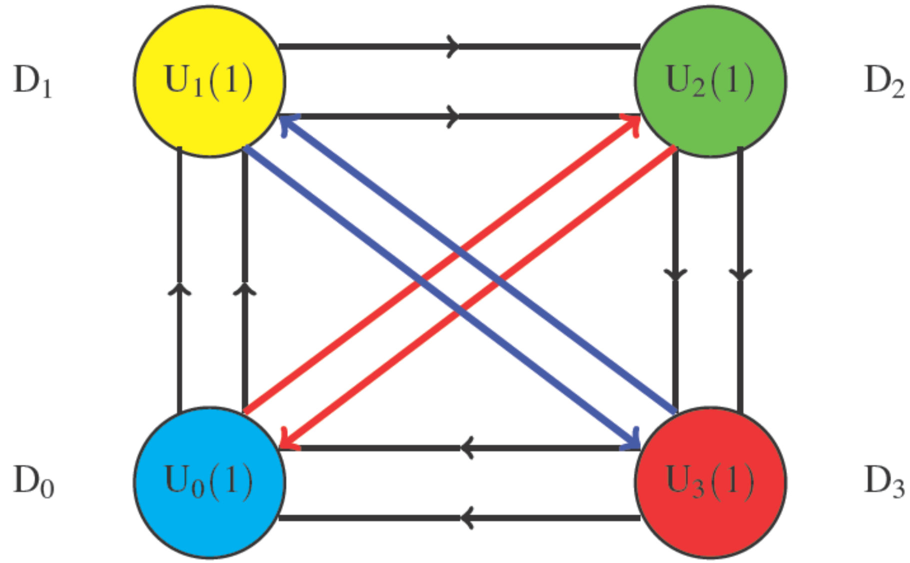
$$[\mathbb{F}_\Gamma, \mathbb{F}_\Gamma] \subset \mathbb{F}_\Gamma \quad ; \quad [\mathbb{F}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{K}_\Gamma \quad ; \quad [\mathbb{K}_\Gamma, \mathbb{K}_\Gamma] \subset \mathbb{F}_\Gamma$$

$$\mathcal{D}_\Gamma = \text{Orbit}_{\mathcal{F}_\Gamma}(\exp[\mathbb{K}_\Gamma] \cdot L_\Gamma)$$

$$\mu^{-1}(\zeta) /\!\!/ \mathcal{F}_\Gamma = \left\{ Z \in \exp[\mathbb{K}_\Gamma] \cdot L_\Gamma \parallel \begin{array}{ll} \mu_I(Z) = 0 & \text{if } f_I \notin \mathfrak{Z} \\ \mu_I(Z) = \zeta_I & \text{if } f_I \in \mathfrak{Z} \end{array} \right\}$$

The $\mathbb{C}^3/\mathbb{Z}_4$ model $A^4 = \mathbf{e}.$

Conj. Class	Matrix	age-vector	age	name
Id	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{4}(0,0,0)$	0	null class
$\mathcal{Q}(A)$	$\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\frac{1}{4}(1,1,2)$	1	junior class
$\mathcal{Q}(A)^2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\frac{1}{4}(2,2,0)$	1	junior class
$\mathcal{Q}(A)^3$	$\begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\frac{1}{4}(3,3,2)$	2	senior class



$$\mathcal{Q}(A) = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this case a complete study was done. The resolved variety $Y_{[3]}^{\mathbb{Z}_4}$ is the total space of the canonical bundle on the second Hirzebruch surface:

$$Y_{[3]}^{\mathbb{Z}_4} = \text{tot}\left(K\left[\mathbb{F}_2\right]\right)$$

Hirzebruch surfaces

Let (U,V) be the homogeneous complex coordinates of a $\mathbb{P}^1$ projective space and (X,Y,Z) the homogeneous complex coordinates of a $\mathbb{P}^2$ projective space.

Definition 5.1. *The n-th Hirzebruch surface $\mathbb{F}_n$ is defined as the locus cut out in $\mathbb{P}^1 \times \mathbb{P}^2$ by the following equation of degree $n+1$:*

$$0 = \mathscr{P}_n(U,V,X,Y,Z) = XU^n - YV^n$$

A theorem in algebraic geometry states that the second Hirzebruch surface does not admit any Kaehler Einstein surface.

Topologically all even degree Hirzebruch surfaces are the product of two spheres.

$$\mathbb{F}_{2n} \sim \mathbb{S}^2 \times \mathbb{S}^2$$

In its complex structure the second Hirz. Surf. is a $\mathbf{P^1}$ - bundle over $\mathbf{P^1}$

$$\mathbb{F}_2 \xrightarrow{\pi} \mathbb{P}^1_b \quad ; \quad \forall p \in \mathbb{P}^1_b \quad \pi^{-1}(p) = \mathbb{P}^1_f$$

Double Fibration

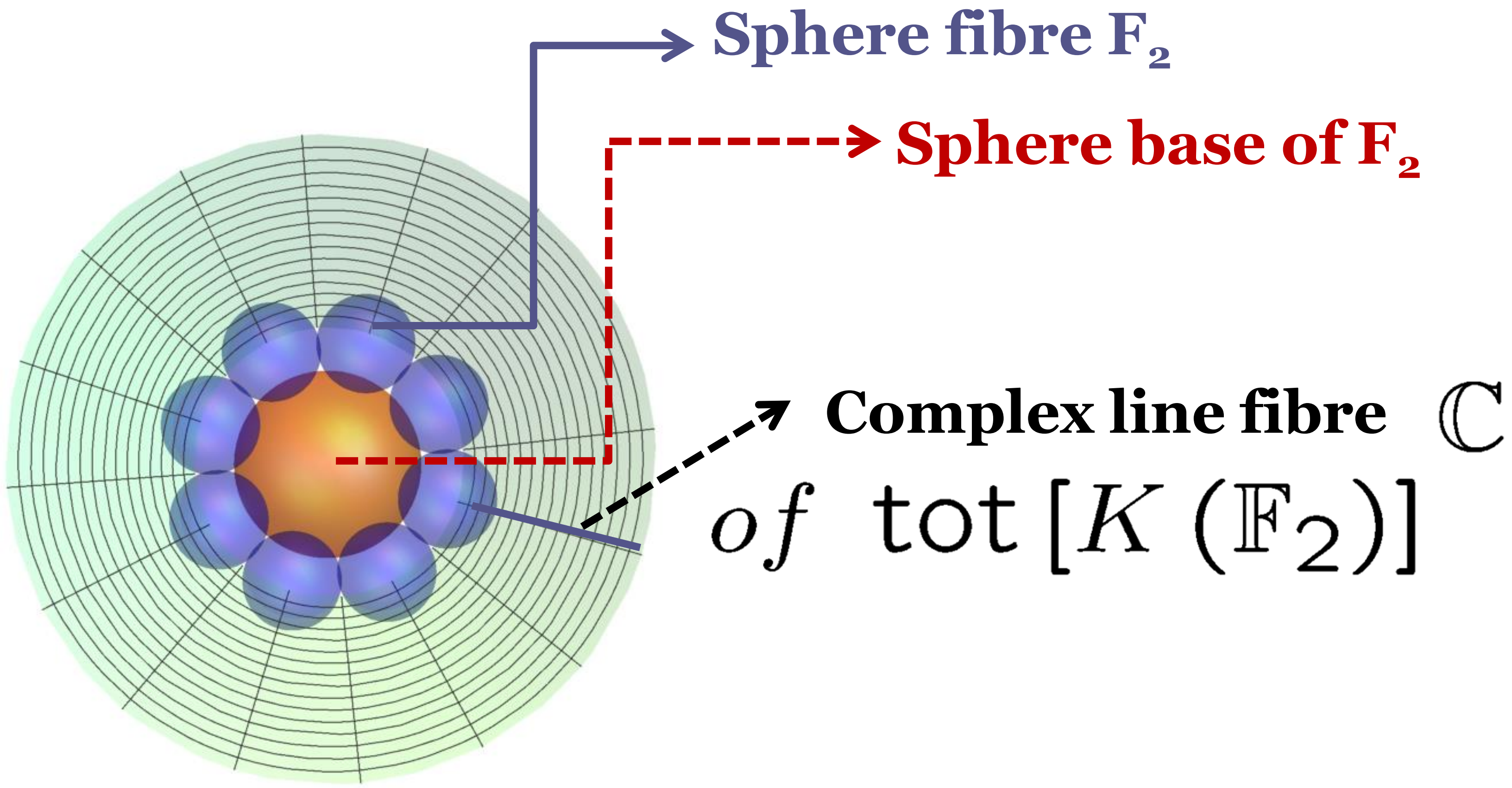
$$Y^{\mathbb{Z}_4}_{[3]} \sim \mathscr{M}_6 = \mathrm{tot}\left(K\left[\mathscr{M}_B\right]\right)$$

$$\mathscr{M}_6 \xrightarrow{\pi_1} \mathscr{M}_B \xrightarrow{\pi_2} \mathbb{P}^1 \sim \mathbb{S}^2$$

How general is this scheme?

The transverse space to the brane M_6 is non compact and smooth. It is a line bundle over a compact 4-manifold M_4 that is the compact exceptional divisor of the resolution, namely shrinks down to a the fixed point in the blowdown map. The exceptional divisor is itself a fibre-bundle.

Conceptual image of the resolved manifold



Predictions from Ito-Reid Theorem

age-vectors

$$c_1 = \frac{1}{4}\{1,1,2\}$$ **junior compact**

$$c_2 = \frac{1}{4}\{2,2,0\}$$ **junior non compact**

$$c_3 = \frac{1}{4}\{3,3,2\}$$ **senior**

Poincaré duality

since we have a compact support (1,1)-cocycle there must be also a (2,2)-cocycle

Here we have a complete illustration. $\mathbf{Z}_4$ has 3 non trivial irreps hence there are three tautological bundles and three $\omega^{1,1}$ closed forms. Yet we expect only two 2-cycles in homology since we have only 2 junior classes. In the ***correspondence line-bundles / divisors*** only ***one compact divisor*** and ***one non compact one***.There is a linear relation between the cohomology classes of the three $\omega^{1,1}$ closed forms.

Toric coordinates

Four open dense charts

$$
\begin{aligned}
\{u,v,w\}_1 &= \left\{ \frac{x}{y} \,,\, \frac{y^2}{z} \,,\, z^2 \right\} \\
\{u,v,w\}_2 &= \left\{ \frac{y}{x} \,,\, \frac{x^2}{z} \,,\, z^2 \right\} \\
\{u,v,w\}_3 &= \left\{ \frac{y}{x} \,,\, \frac{z}{x^2} \,,\, x^4 \right\} \\
\{u,v,w\}_4 &= \left\{ \frac{x}{y} \,,\, y^4 \,,\, \frac{z}{y^2} \right\}
\end{aligned}
$$

The toric construction leads to derive **four dense coordinate patches** with precise transition functions from one to the other.

$$
\begin{array}{llll}
\text{Chart } X_{\sigma_1} & x \to u\sqrt{v}\sqrt[4]{w} \,, & y \to \sqrt{v}\sqrt[4]{w} \,, & z \to \sqrt{w} \\
\text{Chart } X_{\sigma_2} & x \to \sqrt{v}\sqrt[4]{w} \,, & y \to u\sqrt{v}\sqrt[4]{w} \,, & z \to \sqrt{w} \\
\text{Chart } X_{\sigma_3} & x \to \sqrt[4]{w} \,, & y \to u\sqrt[4]{w} \,, & z \to v\sqrt{w} \\
\text{Chart } X_{\sigma_4} & x \to u\sqrt[4]{v} \,, & y \to \sqrt[4]{v} \,, & z \to \sqrt{vw}
\end{array}
$$

We use the first of these open patches

The model $Y^{\mathbb{Z}_4}_{[3]} \xrightarrow{bd} \frac{\mathbb{C}^3}{\mathbb{Z}_4}$

Fayet Iliopoulos parameters

The moment map equations

$$\begin{pmatrix} -\frac{\left(X_1^2-X_3^2\right)\left(U\left(X_2^2+1\right)+\Sigma X_1X_3\right)}{X_1X_2X_3} \\ \frac{\Sigma\left(X_2^3+X_2-X_1X_3\left(X_1^2+X_3^2\right)\right)}{X_1X_2X_3} \\ -\frac{\left(X_2^2-1\right)\left(U\left(X_1^2+X_3^2\right)+\Sigma X_2\right)}{X_1X_2X_3} \end{pmatrix} = \begin{pmatrix} \zeta_3-\zeta_1 \\ -\zeta_1+\zeta_2-\zeta_3 \\ -\zeta_2 \end{pmatrix}$$

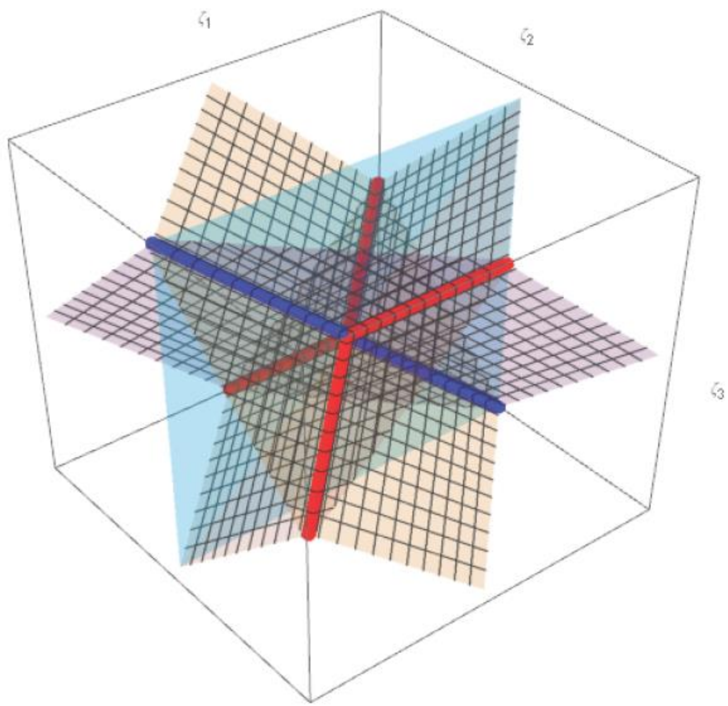
$$\Sigma = \sqrt[4]{|w|^2}\sqrt{(|u|^2+1)^2\,|v|^2} \quad ; \quad U = \sqrt{|w|^2}$$

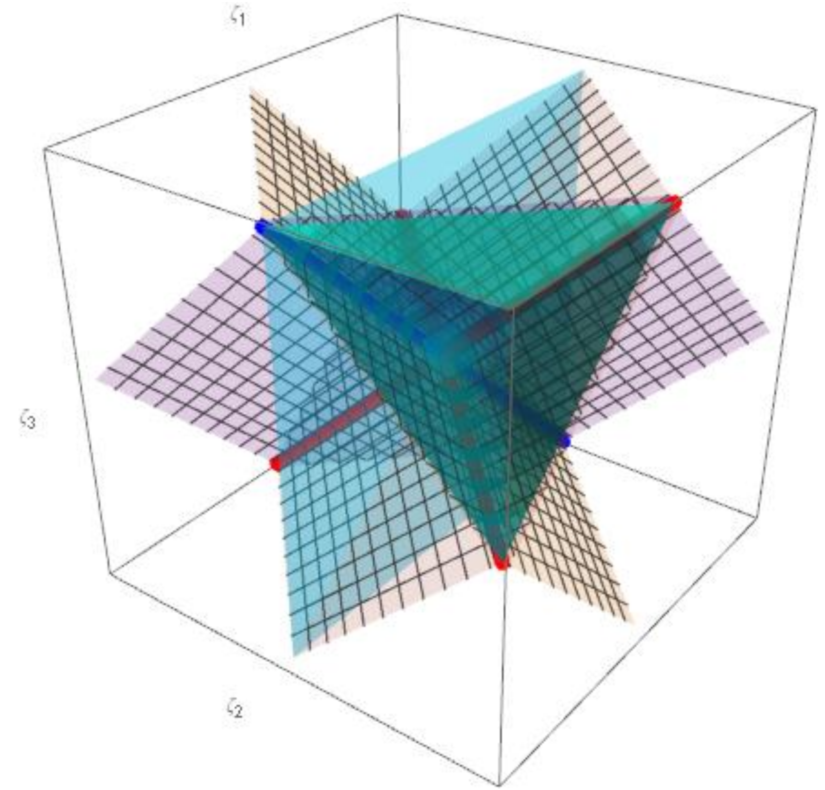
Kaehler potential of HKLR

$$\mathcal{K}_{HKLR} = \mathcal{K}_0 + \zeta_I \mathfrak{C}^{IJ}\log\left(X_J\right)$$

$$\mathcal{K}_0 = \frac{U\left(X_2^2+1\right)\left(X_1^2+X_3^2\right)+\Sigma\left(X_2^3+X_2+X_1X_3\left(X_1^2+X_3^2\right)\right)}{X_1X_2X_3}$$

Chamber structure





Inside the chambers the M_6 is always the canonical bundle on F_2. On walls and edges partial degenerations can occur.

Reduction to the Exceptional Divisor

$$\varpi \equiv (1+|u|^2)^2 |v|^2 \quad ; \quad \lambda \equiv \sqrt{|w|} \quad ; \quad \sigma \equiv |v| \quad ; \quad \delta \equiv \frac{\sqrt{\varpi}}{\sigma}$$

$$X_1 \to T_1 \sqrt{\frac{\lambda}{\sigma}} \quad , \quad X_2 \to \lambda^2 T_2 \quad , \quad X_3 \to T_3 \sqrt{\frac{\lambda}{\sigma}}$$

In the limit $\lambda \to 0$ the moment map equations reduce to solvable ones

$$\begin{pmatrix} T_1\left((\zeta_3-\zeta_1)\,\sigma T_2 T_3 + T_3^3\sqrt{\varpi}\right) - \sigma T_1^2 + \sigma T_3^2 + T_3 T_1^3\left(-\sqrt{\varpi}\right) \\ \sigma T_2\left(\sigma\sqrt{\varpi} - (\zeta_1-\zeta_2+\zeta_3)\,T_1 T_3\right) - T_1 T_3\left(T_1^2+T_3^2\right)\sqrt{\varpi} \\ \sigma\left(-\zeta_2 T_2 T_3 T_1 + \sigma T_2\sqrt{\varpi} + T_1^2 + T_3^2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and the result is an explicit Kaehler potential for the Kaehler metric on the exceptional divisor $\mathbf{F_2}$

Choosing a line inside a chamber

$$\zeta_1 = \zeta_3 = s > 0 \quad ; \quad \zeta_2 = s(2+\alpha) \quad ; \quad \alpha > 0$$

$$\begin{aligned}\mathcal{K}(\varpi) \; = \; & \frac{1}{2}\left(\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+3\alpha-\varpi+4\right) \\ & +2(\alpha+1)\log\left(\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi^2+8\varpi}+3\alpha+\varpi+4}{2\alpha^2+6\alpha+4}\right) \\ & -2\alpha\log\left(\frac{\sqrt{\frac{\sqrt{\alpha^2+6\alpha\varpi+\varpi(\varpi+8)}+\alpha+\varpi}{(\alpha+2)\sqrt{\varpi}}}}{\sqrt{2}}\right)\end{aligned}$$

Kaehler potential of an explicit Kaehler metric on the F_2 surface

AMSY formalism

$\mathscr{K}(|z_1|, .., |z_n|)$, where $z_i = e^{x_i + i\Theta_i}$

Toric invariance

$\mu^i = \partial_{x_i}\mathscr{K}$

moments

$$G(\mu_i) = \sum_i^n x_i \mu^i - \mathscr{K}(|z_1|, .., |z_n|)$$

Symplectic potential obtained from the Legendre transform

$$G_{ij} = \frac{\partial^2}{\partial\mu^i \partial\mu^j} G(\mu)$$

Hessian

Kaehler 2-form

$$\mathbb{K} = \sum_{i=1}^n \mathrm{d}\mu^i \wedge \mathrm{d}\Theta_i$$

$$ds^2_{symp} = \mathbf{G}_{ij} d\mu^i d\mu^j + \mathbf{G}_{ij}^{-1} d\Theta^i d\Theta^j$$

The metric

A family of cohomogeneity one Kaehler metrics on 4-dim manifolds

$$G_{\mathscr{M}_B} = G_0(\mathfrak{u},\mathfrak{v}) + \mathscr{D}(\mathfrak{v})$$

$$G_0(\mathfrak{u},\mathfrak{v}) = \left(\mathfrak{v} - \frac{\mathfrak{u}}{2}\right)\log(2\mathfrak{v}-\mathfrak{u}) + \frac{1}{2}\mathfrak{u}\log(\mathfrak{u}) - \frac{1}{2}\mathfrak{v}\log(\mathfrak{v})$$

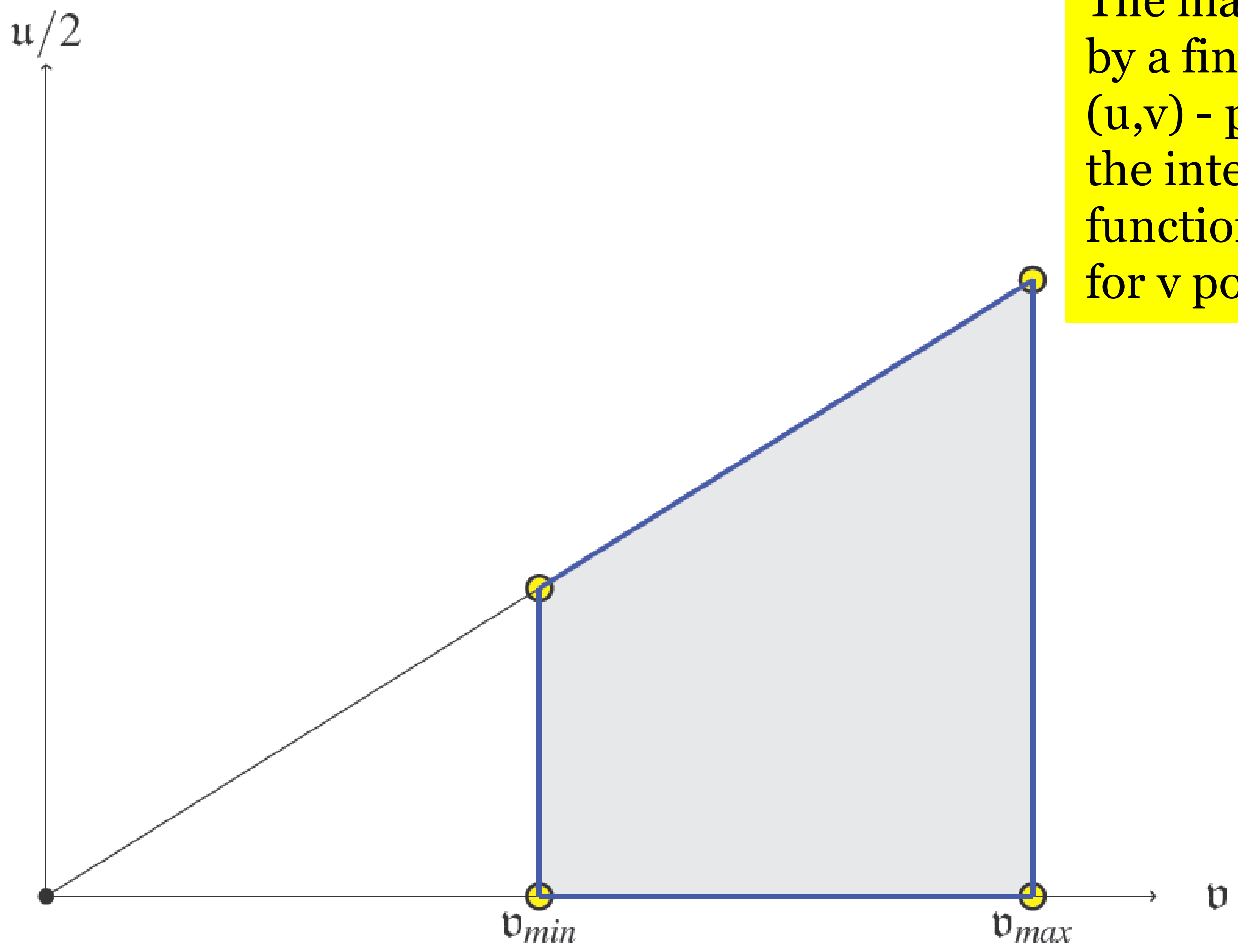
This is the equivalent for the symplectic potential of the dependence of the Kaehler potential only on ϖ

$$ds^2_{\mathcal{M}_B} = \frac{d\mathfrak{v}^2}{\mathcal{FK}(\mathfrak{v})} + \mathcal{FK}(\mathfrak{v})\left[d\phi(1-\cos\theta) + d\tau\right]^2 + \mathfrak{v}\underbrace{\left(d\phi^2\sin^2\theta + d\theta^2\right)}_{\mathbb{S}^2\ metric}$$

$$\mathscr{F}\mathscr{K}(\mathfrak{v}) = \frac{2\mathfrak{v}}{2\mathfrak{v}\mathscr{D}''(\mathfrak{v})+1}$$

$$\mathfrak{u} \to (1-\cos\theta)\,\mathfrak{v}$$

The polytope



The manifold is described by a finite region of the (u,v) - plane defined by the interval in which the function FK(v) is positive for v positive.

SU(2)×U(1) isometry

Against the following transformations

$$\text{if } \mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(2) \quad \text{then} \quad \mathbf{g}(u,v) = \left(\frac{au+b}{cu+d}, \quad v\,(cu+d)^2 \right)$$

$$\text{if } \mathbf{g} = \exp(i\theta_1) \in \mathrm{U}(1) \quad \text{then} \quad \mathbf{g}(u,v) = (u, \quad \exp(i\theta_1)\,v)\,.$$

The object that follows is invariant

$$\varpi \equiv (1 + |u|^2)^2\,|v|^2$$

If the Kaehler potential is a function only of ϖ , then the corresponding Kaehler metric is isometric with respect to SU(2) × U(1)

Inverse Legendre transform

$$\mathcal{K}_0 = x_u\,\mathfrak{u} + x_v\,\mathfrak{v} - G_{\mathcal{M}_B}(\mathfrak{u},\mathfrak{v})$$

$$x_u = \partial_{\mathfrak{u}} G_{\mathcal{M}_B}(\mathfrak{u},\mathfrak{v}) \quad ; \quad x_v = \partial_{\mathfrak{v}} G_{\mathcal{M}_B}(\mathfrak{u},\mathfrak{v})$$

$$x_u = \frac{1}{2}\left(\log(\mathfrak{u}) - \log(2\mathfrak{v}-\mathfrak{u})\right) \quad ; \quad x_v = \mathcal{D}'(\mathfrak{v}) + \log(2\mathfrak{v}-\mathfrak{u}) - \frac{1}{2}\log(\mathfrak{v}) + \frac{1}{2}$$

The entire structure of the metric is codified in the function $\mathcal{D}(\mathfrak{v})$

$$\varpi = \Omega(\mathfrak{v}) \equiv 4\mathfrak{v}\exp\left[2\,\partial_{\mathfrak{v}}\mathcal{D}(\mathfrak{v}) + 1\right]$$

$$\mathcal{K}_0 = \mathfrak{v}\left(\mathcal{D}'(\mathfrak{v}) + \frac{1}{2}\right) - \mathcal{D}(\mathfrak{v})$$

If the Kaehler potential is given as a function of the moment map one can always reconstruct the function $\mathcal{D}(\mathfrak{v})$, however not always one is able to invert the function Ω and give the momentum $\mathfrak{v}$ in terms of the invariant ϖ

The notable cases in the family are the Kronheimer metric on F_2, a degenerate case that we find on the walls $WP_{1,1,2}$, a new family the KE metrics, recently found by us and actually discovered to be part of a 4 parameter family found by Calabi 50 years ago but in different coordinates

$\mathscr{F}\mathscr{K}^{\mathbb{F}_2}_{Kro}(\mathfrak{v}) = \frac{(1024\mathfrak{v}^2-81\alpha^2)(32\mathfrak{v}-9(3\alpha+4))}{16(81\alpha^2+1024\mathfrak{v}^2-576(3\alpha+4)\mathfrak{v})}$	$\mathfrak{v}_{min} = \frac{9\alpha}{32}$	$\mathfrak{v}_{max} = \frac{9}{32}(3\alpha+4)$	$\alpha > 0$
$\mathscr{F}\mathscr{K}^{\mathrm{WP}_{[1,1,2]}}_{Kro}(\mathfrak{v}) = \frac{\mathfrak{v}(8\mathfrak{v}-9)}{4\mathfrak{v}-9}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \frac{9}{8}$	$\alpha = 0$
$\mathscr{F}\mathscr{K}_{ext}(\mathfrak{v}) = \frac{-\mathscr{A}+8\mathscr{C}\mathfrak{v}^3-16\mathscr{D}\mathfrak{v}^4+4\mathfrak{v}^2-2\mathfrak{v}\mathscr{B}}{4\mathfrak{v}}$	$\mathfrak{v}_{min} = \lambda^r_1$	$\mathfrak{v}_{max} = \lambda^r_2$	$0 < \lambda^r_1 < \lambda^r_2$
$\underbrace{\mathscr{F}\mathscr{K}^{KE}(\mathfrak{v})}_{\mathscr{B}=\mathscr{D}=0} = -\frac{(\mathfrak{v}-\lambda_1)(\mathfrak{v}-\lambda_2)(\lambda_2\mathfrak{v}+\lambda_1(\lambda_2+\mathfrak{v}))}{(\lambda_1^2+\lambda_2\lambda_1+\lambda_2^2)\mathfrak{v}}$	$\mathfrak{v}_{min} = \lambda_1$	$\mathfrak{v}_{max} = \lambda_2$	$0 < \lambda_1 < \lambda_2$
$\mathscr{F}\mathscr{K}^{KE}_0(\mathfrak{v}) = \frac{\mathfrak{v}(\lambda_2-\mathfrak{v})}{\lambda_2}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \lambda_2$	$\lambda_2 > 0$
$\mathscr{F}\mathscr{K}^{cone}(\mathfrak{v}) = \mathfrak{v}$	$\mathfrak{v}_{min} = 0$	$\mathfrak{v}_{max} = \infty$	
$\mathscr{F}\mathscr{K}^{\mathbb{F}_2}_{ext}(\mathfrak{v}) = \frac{(a-\mathfrak{v})(b-\mathfrak{v})\left(a^2(3b-\mathfrak{v})+a\left(b^2+4b\mathfrak{v}+3\mathfrak{v}^2\right)+b\mathfrak{v}(b+\mathfrak{v})\right)}{\mathfrak{v}(a^3+3a^2b-3ab^2-b^3)}$	$\mathfrak{v}_{min} = a$	$\mathfrak{v}_{max} = b$	$b > a > 0$

General structure of the Kaehler potential for the distinguished cases of the SU(2)×U(1) metric family and the polytope

$$\mathscr{K}(\mathfrak{v}) = \left(\frac{1}{2} + k_0 + \sum_{i=1}^{4} k_i\right)\mathfrak{v} + \sum_{i=1}^{4} k_i \lambda_i \log[v - \lambda_i]$$

The metrics that arise in the quotient C^3/Z_4 on the exceptional compact divisor

$\lambda_0 = 0$	$\lambda_1 = -\frac{9\alpha}{32}$	$\lambda_2 = \frac{9\alpha}{32}$	$\lambda_3 = \frac{9(4+3\alpha)}{32}$	$\lambda_4 =$ arbirary
$k_0 = -\frac{1}{2}$	$k_1 = \frac{1}{2}$	$k_2 = \frac{1}{2}$	$k_3 = \frac{1}{2}$	$k_4 = 0$

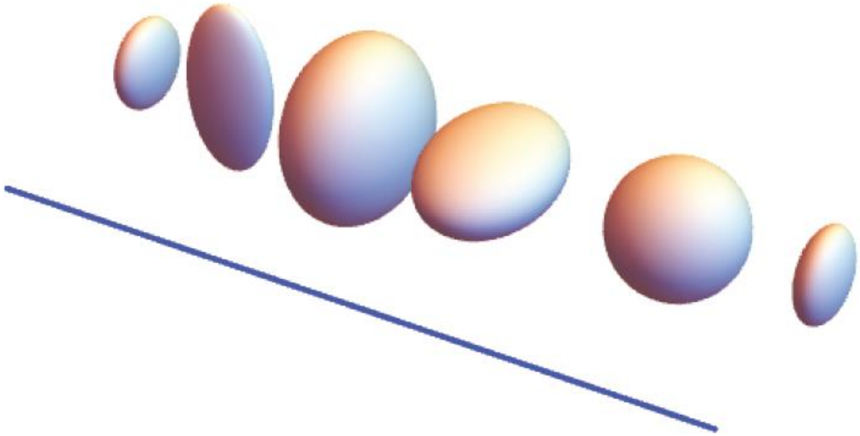
The Kaehler Einstein metrics

$\lambda_0 = 0$	$\lambda_1 = a$	$\lambda_2 = b$	$\lambda_3 = \frac{ab}{a+b}$	$\lambda_4 =$ arbirary
$k_0 = -\frac{1}{2}$	$k_1 = -\frac{a^2+ab+b^2}{a^2+ab-2b^2}$	$k_2 = -\frac{a^2+ab+b^2}{-2a^2+ab+b^2}$	$k_3 = \frac{a^2+ab+b^2}{2a^2+5ab+2b^2}$	$k_4 = 0$

For these metrics it is not clear, yet, whether we can derive them from a McKay quotient C^3/Γ and from which Γ.

They are very much interesting because we can use the Calabi Ansatz and construct explicitly the Ricci flat metric on their canonical bundle. Hence we have the full D3 brane solution!

What are the KE manifolds geometrically?



The two real manifolds defined by the restriction to the dense chart $\mathfrak{u}, \mathfrak{v}, \phi, \tau$, of the surface $\mathbb{F}_2$ and of the manifold $\mathcal{M}_B^{KE}$ are fully analogous. Cutting the compact four manifold into $\mathfrak{v}$ = const slices we always obtain the same result, namely a three manifold $\mathcal{M}_3$ with the structure of a circle fibration on $\mathbb{S}^2$:

$$\mathcal{M}_B \supset \mathcal{M}_3 \xrightarrow{\pi} \mathbb{S}^2 \quad ; \quad \forall p \in \mathbb{S}^2 \quad \pi^{-1}(p) \sim \mathbb{S}^1$$

Explicit form of the metric at fixed moment map v

$$ds^2_{\mathscr{M}_3} = \mathfrak{v}\left(d\phi^2 \sin^2\theta + d\theta^2\right) + \mathscr{F}\mathscr{K}(\mathfrak{v})\left[d\phi(1-\cos\theta) + d\tau\right]^2$$

The fixed parameter $\mathfrak{v}$ plays the role of the squared radius of the sphere $\mathbb{S}^2$ while $\sqrt{\mathscr{F}\mathscr{K}(\mathfrak{v})}$ weights the contribution of the circle fibre defined over each point $p \in \mathbb{S}^2$. At the endpoints of the intervals $\mathscr{F}\mathscr{K}(\mathfrak{v}_{min}) = \mathscr{F}\mathscr{K}(\mathfrak{v}_{max}) = 0$ the fibre shrinks to zero.

The difference between F_2 and the KE manifolds is simply what happens in the extremal points. For F_2 there is no deficit angle. For KE there is a deficit angle and therefore there are two conical singularities (different one from the other)

$$ds^2 = \frac{d\mathfrak{v}^2}{\mathscr{F}\mathscr{K}(\mathfrak{v})} + \mathscr{F}\mathscr{K}(\mathfrak{v})\, d\tau^2$$

$$\mathscr{F}\mathscr{K}(\mathfrak{v}) = \mathscr{F}\mathscr{K}'(\lambda)(\mathfrak{v}-\lambda) + \mathscr{O}((\mathfrak{v}-\lambda)^2)$$

$$\begin{array}{lclcl} \text{KE} & : & \mathscr{F}\mathscr{K}'_{KE}(\lambda_1) = \frac{(\lambda_2-\lambda_1)(\lambda_1+2\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2} & ; & \mathscr{F}\mathscr{K}'_{KE}(\lambda_2) = \frac{(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2)}{\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2}, \\ \mathbb{F}_2 \text{ Kronheimer} & : & \mathscr{F}\mathscr{K}'_{\mathbb{F}_2|Kro}\left(\frac{9\alpha}{32}\right) = 2 & ; & \mathscr{F}\mathscr{K}'_{\mathbb{F}_2|Kro}\left(\frac{9(3\alpha+4)}{32}\right) = -2 \end{array}$$

$$ds^2 = dr^2 + \beta^2 r^2 d\tau^2$$

$$r = 2\sqrt{\frac{\mathfrak{v}-\lambda}{\mathscr{F}\mathscr{K}'(\lambda)}}, \quad \beta = \frac{|\mathscr{F}\mathscr{K}'(\lambda)|}{2}$$

If β is not 1 we have a deficit angle and therefore a conical singularity

Conclusion

There are three main directions of development

- Try to implement the Kaehler quotient procedure directly at the level of the AMSY formalism in order to clarify the relation between the group Γ and the polytopes
- Explore instances of non abelian Γ in full detail.
- Consider deformations of the cubic superpotential and study their classification and systematics (I did not alk about that. It is work in progress with Massimo Bianchi).

Thank you very much for your attention