Supergravity Black Holes and Nilpotent Orbits

PIETRO FRÉ
TORINO UNIVERSITY, ITALY
AND EMBASSY OF ITALY IN THE RUSSIAN FEDERATION

Talk given at the Sternberg Institute
MGU, Moscow
March 16th 2011

Based on work in collaboration with A.S. Sorin & M. Trigiante

Newtonian Gravity

Two bodies of mass M and m at a distance R attract each other with a force

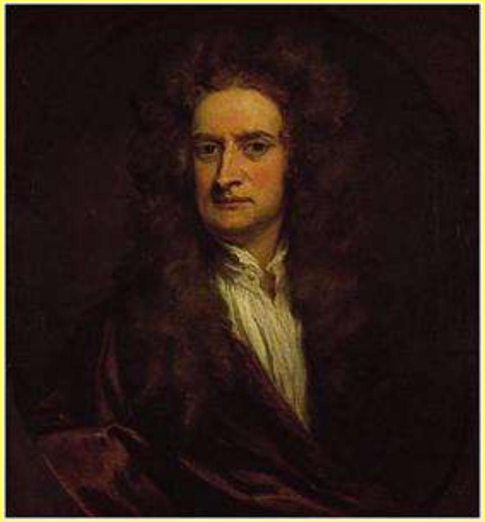
$$F = -G\frac{mM}{R^2}$$

where G is Newton's constant

$$G = (6,67259 \pm 0.00085) \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$$

From this formula we work out the ***escape velocity*** (namely the minimal velocity that a body must have in order to be able to escape from the surface of a star having radius R and mass M)

$$v_f = \sqrt{\frac{2GM}{R}}$$



Pianeta	Velocità di fuga
Mercurio	4,435 km/s
Venere	10,4 km/s
Terra	11,2 km/s
Marte	5,04 km/s
Giove	59,5 km/s
Saturno	35,6 km/s
Urano	21,3 km/s
Nettuno	23,3 km/s
Plutone	1,3 km/s

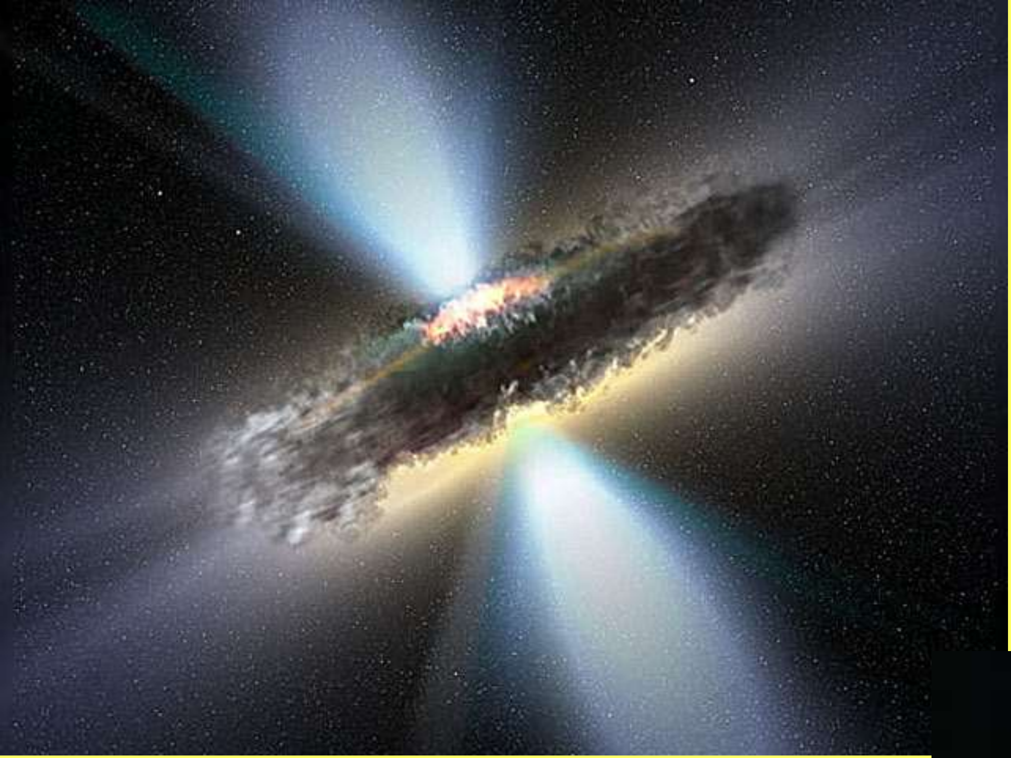
Laplace 1796

In his ***Exposition du System de Monde***, Laplace foresaw the possibility that a **celestial body** with ***radius R*** might have a ***mass M*** **so big** that the correspondent escape velocity is larger than the speed of light:

$$v_f = \sqrt{\frac{2GM}{R}} > c$$

In this case **the celestial body** would be **invisible**.....

Indeed no light-signal could emerge from it and reach us: BLACK HOLE.



<u>True astrophysical Black Holes are not the main concern in this talk</u>

Supermassive Black Holes (10^6 solar masses) are the hidden engines of Galaxies and Quasars

Black holes are revealed by the observation of their accretion disk and of the flares orthogonal to the accretion disk plane.

The Black Holes we deal with are solitons of String Theory…..! *No who knows…?*

KARL SCHWARZSCHILD

- 1873 – 1916 (He was born in Frankurt am Mein in a well to do Jewish family)

- Very young determined orbits of binary stars

- Since 1900 Director of the Astronomical Observatory of Gottingen (the hottest point of the world for Physics and Mathematics at that time and in subsequent years)

- Famous scientist and member of the Prussian Academy of Sciences in 1914 he enrolled as a volounteer in the German Army and went to war first on the western and then on the eastern front against Russia.

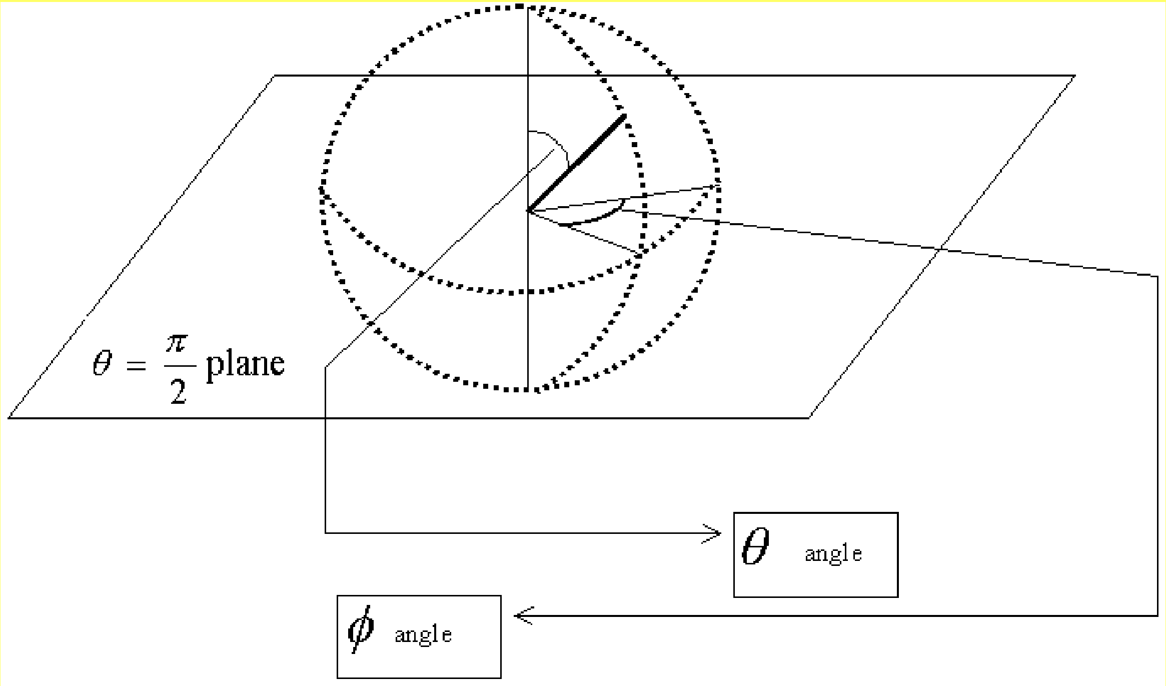
- At the front in 1916 he wrote two papers. One containing quantization rules discovered by him independentely from Sommerfeld. The second containing Schwarzschild solution of GR. At the front he had learnt GR two months before reading Einstein's paper.

- Einstein wrote to Schwarschild : ….*I did not expect that one could formulate the exact solution of the problem in such a simple way….*

- Few months later Schwarschild died from an infection taken at the front

1916

Fundamental solution: the Schwarzschild metric (1916)



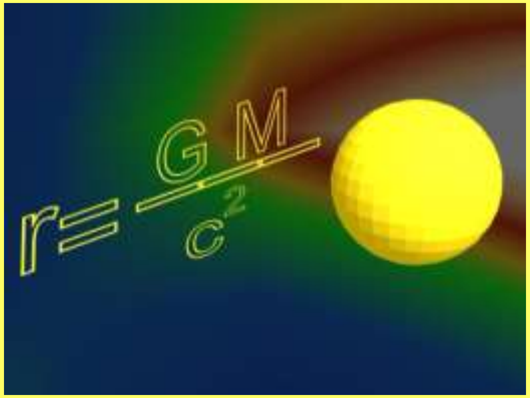
Using standard polar coordinates plus the time coordinate **t**

$$ds^2 = -g_{\mu\nu}(x)dx^\mu\, dx^\nu$$

$$ds^2 = -\left(1-\frac{2r}{R}\right)dt^2 + \left(1-\frac{2r}{R}\right)^{-1} dR^2 + R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Is the most general static and spherical symmetric metric

The Schwarzschild metric has a singularity at the Schwarzschild radius



It took about 50 years before its true interpretation was found.

Kruskal

In the meantime another solution was found

$$ds^2_{RN} = -\left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right) dt^2 + \left(1-\frac{2m}{r}+\frac{q^2}{r^2}\right)^{-1} dr^2$$
$$+r^2\left(d\theta^2 + \sin^2 d\phi^2\right)$$

$$F = -\frac{\sqrt{2}\,q}{\kappa}\,\frac{1}{r^2}\,dt \wedge dr$$

This is the **Reissner Nordstrom** solution which describes a spherical object with both mass m and charge q

The Reissner Nordstrom solution

Hans Jacob Reissner was a German aeronautical engineer with a passion for mathematical physics. He solved Einstein equations with an electric field in 1916. Later he emigrated to the USA and was professor in Illinois and in Brooklin

Gunnar Nordstrom was a finnish theoretical physicist who worked in the Netherlands at Leiden in Ehrenfest's Institute. In 1918 he solved Eisntein's equation for a spherical charged body extending Reissner's solution for a point charge.

Hans Jacob Reissner (1874-1967) & wife in 1908

Gunnar Nordstrom (1881-1923)

THIS EARLY SOLUTION HAS AN IMPORTANT PROPERTY which is the tip of an iceberg of knowledge….*when m=q….*

The first instance of extremal Black Holes

The Reissner Nordstrom metric:

$$ds^2 = -dt^2\left(1-\frac{2m}{\rho}+\frac{q}{\rho^2}\right)+d\rho^2\left(1-\frac{2m}{\rho}+\frac{q}{\rho^2}\right)^{-1}+\rho^2\,d\Omega^2$$

Has two "horizons" at

$$\rho_\pm = m \pm \sqrt{m^2-q^2}$$

There is a true singularity that becomes "naked" if

$$m \leq |q|$$

It was conjectured a principle named COSMIC CENSORSHIP

$$m \geq |q|$$

It is intimately related to supersymmetry

Extremal Reissner Nordstrom solutions

$$m = |q| \quad ; \quad \rho = r + m \quad ; \quad r^2 = \vec{x} \cdot \vec{x}$$

$$\begin{aligned} ds^2 &= -dt^2 \left(1 + \frac{q}{r}\right)^{-2} + \left(1 + \frac{q}{r}\right)^2 \left(dr^2 + r^2\, d\Omega^2\right) \\ &= -H^{-2}(\vec{x})\, dt^2 + H^2(\vec{x})\, d\vec{x} \cdot d\vec{x} \end{aligned}$$

Harmonic function

$$H(\vec{x}) = \left(1 + \frac{q}{\sqrt{\vec{x} \cdot \vec{x}}}\right)$$

The largest part of new developments in BH physics is concerned with generalizations of this solution and with their deep relations with Supergravity and Superstrings

Historical introduction

1. 1995-96 Discovery of the attraction mechanism in BPS BH.s by Ferrara & Kallosh
2. 1995-96 Discovery of the statistical interpretation of the area of the horizon as counting of string microstates by Strominger.
3. 1997-1999 Extensive study of BPS solutions of supergravity of BH type by means of FIRST ORDER EQUATIONS, following from preservation of SUSY
4. NEW WAVE of interest mid 2000s: also non BPS Black Holes have the attraction mechanism! Also there we can find a fake-superpotential!
5. D=3 approach and NOW INTEGRABILITY!

1995-1998 Ferrara, Kallosh, Strominger

Sergio Ferrara born in Rome, one of the three founders of Supergravi... @CERN, p...

Renata Kallosh graduated from MSU in 1966, prof. in Stanford

...ndrew Strominger
Harvard Professor

The main pioneers of the new Black Hole season

What is a BPS black hole?

- To explain this idea we have to introduce a few basic facts about the supersymmetry algebra…..

Extended SUSY algebra in D=4

$$\left\{\bar{Q}_{A\alpha}\,,\,\bar{Q}_{B\beta}\right\} = \mathrm{i}\,(\mathbf{C}\,\gamma^{\mu})_{\alpha\beta}\,P_{\mu}\,\delta_{AB}\; - \;\mathbf{C}_{\alpha\beta}\,\mathbf{Z}_{AB}$$

$$(A, B = 1, \ldots, 2p)$$

NORMAL FORM of CENTRAL CHARGES

$$\mathbf{Z}_{AB} = \begin{pmatrix} \epsilon Z_1 & 0 & \ldots & 0 \\ 0 & \epsilon Z_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \epsilon Z_p \end{pmatrix} \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A = (a, I) \quad ; \quad a, b, \ldots = 1, 2 \quad ; \quad I, J, \ldots = 1, \ldots, p$$

Rewriting of the algebra

$$\left\{\bar{Q}_{aI|\alpha}\,,\,\bar{Q}_{bJ|\beta}\right\} = \mathsf{i}\,(C\,\gamma^{\mu})_{\alpha\beta}\,P_{\mu}\,\delta_{ab}\,\delta_{IJ} - C_{\alpha\beta}\,\epsilon_{ab} \times \mathbb{Z}_{IJ}$$

Bogomolny Bound

$$M \;\geq\; |\,Z_I| \qquad\qquad \forall Z_I\,,\, I = 1,\ldots,p$$

Reduced supercharges

$$\bar{S}^{\pm}_{aI|\alpha} = \frac{1}{2}\,(\bar{Q}_{aI}\gamma_0 \pm \mathsf{i}\,\epsilon_{ab}\,\bar{Q}_{bI}\,)_{\alpha}$$

$$\begin{cases} \bar{S}^{\pm}_{aI} = \bar{Q}_{bI}\,\mathbb{P}^{\pm}_{ba} \\ \mathbb{P}^{\pm}_{ba} = \dfrac{1}{2}\,(\mathbf{1}\delta_{ba} \pm \mathsf{i}\epsilon_{ba}\gamma_0) \end{cases}$$

$$\left\{\bar{S}^{\pm}_{aI}\,,\,\bar{S}^{\pm}_{bJ}\right\} = \pm\epsilon_{ac}\,C\,\mathbb{P}^{\pm}_{cb}\,(M \mp Z_I)\,\delta_{IJ}$$

BPS states = short susy multiplets

$$(M \pm Z_I)\, |\text{BPS state}, i\rangle = 0$$

$$\bar{S}^{\pm}_{aI}\, |\text{BPS state}, i\rangle = 0$$

Field theory description

$$0 = \delta\text{fermions} = \text{SUSY rule}\,(\text{bosons}, \epsilon_{AI})$$

$$\begin{aligned} \xi^{\mu}\,\gamma_{\mu}\,\epsilon_{aI} &= \mathrm{i}\,\varepsilon_{ab}\,\epsilon^{bI} \quad ; \quad I = 1, \ldots, n_{max} \\ \epsilon_{aI} &= 0 \quad ; \quad I > n_{max} \end{aligned}$$

A lesson taught by RN Black Holes

$$\text{Area}_H = \int_{\rho=\rho_+} \sqrt{g_{\theta\theta}\, g_{\phi\phi}}\, d\theta\, d\phi = 4\pi\, \rho_+^2 = 4\pi \left(m + \sqrt{m^2 - |q|^2}\right)^2$$

$$\frac{\text{Area}_H}{4\pi} = |q|^2$$

For m=|q|

For extremal Black Holes the area of the horizon depends only on the charges

The N=2 Supergravity Theory

$$\mathcal{L}^{(4)} = \sqrt{\det g}\Big[-2R[g] - \frac{1}{6}\partial_{\widehat{\mu}}\phi^a\partial^{\widehat{\mu}}\phi^b h_{ab}(\phi) + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\widehat{\mu}\widehat{\nu}} F^{\Sigma|\widehat{\mu}\widehat{\nu}}\Big] + \frac{1}{2}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, F^{\Lambda}_{\widehat{\mu}\widehat{\nu}} F^{\Sigma}_{\widehat{\rho}\widehat{\sigma}}\epsilon^{\widehat{\mu}\widehat{\nu}\widehat{\rho}\widehat{\sigma}}$$

We have gravity
and
n vector multiplets

→

2 n scalars yielding n complex scalars z^i

and n+1 vector fields $\mathbf{A}^\Lambda$

The matrix $N_{\Lambda\Sigma}$ encodes together with the metric h_{ab} Special Geometry

Special Kahler Geometry

Let $\mathcal{L} \longrightarrow \mathcal{M}$ complex line bundle such that first Chern class equals Kähler form K. Let $\mathcal{SV} \longrightarrow \mathcal{M}$ be a holomorphic flat vector bundle of rank $2n+2$ with structural group $\mathrm{Sp}(2n+2, \mathrm{R})$

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \ldots, n$$

symplectic section

$$\mathrm{i}\langle \Omega \,|\, \bar{\Omega} \rangle \equiv \mathrm{i}\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \bar{\Omega}$$

$$\mathrm{K} = \frac{\mathrm{i}}{2\pi} \partial \bar{\partial} \log \left(\mathrm{i}\langle \Omega \,|\, \bar{\Omega} \rangle \right)$$

Special Geometry identities

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathcal{K}/2}\Omega = e^{\mathcal{K}/2}\begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}$$

$$U_i = \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i \mathcal{K}\right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix}$$

$$\bar{U}_{i^\star} = \nabla_{i^\star}\bar{V} = \left(\partial_{i^\star} + \frac{1}{2}\partial_{i^\star}\mathcal{K}\right)\bar{V} \equiv \begin{pmatrix} \bar{f}_{i^\star}^\Lambda \\ \bar{h}_{\Sigma|i^\star} \end{pmatrix}$$

$$\nabla_i V = U_i$$

$$\nabla_i U_j = \mathrm{i} C_{ijk} g^{k\ell^\star} U_{\ell^\star}$$

$$\nabla_{i^\star} U_j = g_{i^\star j} V$$

$$\nabla_{i^\star} V = 0$$

The matrix $N_{\Lambda\Sigma}$

the two $(n+1) \times (n+1)$ vectors

$$f_I^{\Lambda} = \begin{pmatrix} f_i^{\Lambda} \\ \bar{L}^{\Lambda} \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \bar{M}_{\Lambda} \end{pmatrix}$$

$$\bar{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^{I}_{\ \Sigma}$$

When the special manifold is a symmetric coset ..

$$\mathcal{SK}_n = \frac{\mathsf{U}_{D=4}}{\mathsf{H}_{D=4}}$$

$$U_{D=4} \ni \mathbb{L}(\phi) \mapsto \left(\begin{array}{c|c} A(\phi) & B(\phi) \\ \hline C(\phi) & D(\phi) \end{array}\right) \in \mathsf{Sp}(2\mathsf{n}+2,\mathsf{R})$$

Symplectic embedding

$$\begin{aligned} \mathbf{f} &= \frac{1}{\sqrt{2}}\left(A(\phi) - \mathrm{i}\, B(f)\right) \\ \mathbf{h} &= \frac{1}{\sqrt{2}}\left(C(\phi) - \mathrm{i}\, D(f)\right) \\ \overline{\mathcal{N}}(\phi) &= \mathbf{h}\,\mathbf{f}^{-1} \end{aligned}$$

Dimensional Reduction to D=3

THE C-MAP

D=4 SUGRA with SK_n → D=3 σ-model on Q_{4n+4}

$$ds^2_Q = \frac{1}{4}\left[dU^2 + g_{i\bar{\jmath}}dz^i d\bar{z}^{\bar{\jmath}} + e^{-2U}(da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z})^2 \mp 2e^{-U} d\mathbf{Z}^T \mathcal{M}_4(z,\bar{z})\, d\mathbf{Z}\right]$$

Space red. / Time red.
Cosmol. / Black Holes

$$\underbrace{\{U,a\}}_{2} \bigcup \underbrace{\{z^i\}}_{2n} \bigcup \underbrace{\mathbf{Z} = \{Z^\Lambda, Z_\Sigma\}}_{2n+2}$$

4n + 4 coordinates

Gravity **scalars** **From vector fields**

$$\mathcal{M}_4 = \left(\begin{array}{c|c} \mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} \\ \hline \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1} & \mathrm{Im}\mathcal{N} + \mathrm{Re}\mathcal{N}\,\mathrm{Im}\mathcal{N}^{-1}\mathrm{Re}\mathcal{N} \end{array}\right)$$

When homogeneous symmetric manifolds

$$\frac{U_{D=4}}{H_{D=4}} \rightarrow \frac{U_{D=3}}{H_{D=4}}$$

C-MAP

$$U_{D=3} \supset U_{D=4}$$

General Form of the Lie algebra decomposition

$$\text{adj}(\mathbb{U}_{D=3}) = \text{adj}(\mathbb{U}_{D=4}) \oplus \text{adj}(SL(2,\mathbb{R})_E) \oplus W_{(2,\mathbf{W})}$$

$$[T^a, T^b] = f^{ab}{}_c\, T^c$$
$$[L^x, L^y] = f^{xy}{}_z\, L^z,$$
$$[T^a, W^{iM}] = (\Lambda^a)^M{}_N\, W^{iN},$$
$$[L^x, W^{iM}] = (\lambda^x)^i{}_j\, W^{jM},$$
$$[W^{iM}, W^{jN}] = \epsilon^{ij}\, (K_a)^{MN}\, T^a + \mathbb{C}^{MN}\, k^{ij}_x\, L^x$$

The simplest example $G_{2(2)}$

One vector multiplet

$$\text{adj}\left[\mathfrak{g}_{2(2)}\right] = (\text{adj}\,[\mathfrak{sl}(2,\mathbb{R})_E]\ \mathbf{1}) \oplus (\mathbf{1}\,,\ \text{adj}\,[\mathfrak{sl}(2,\mathbb{R})]) \oplus (\mathbf{2}\,,\ \mathbf{4})$$

$$g_{z\bar{z}} dz\, d\bar{z} = \frac{3}{4} \frac{1}{(\text{Im} z)^2} \partial^\mu z\, \partial_\mu \bar{z}$$ **Poincaré metric**

$$\Omega(z) = \begin{pmatrix} -\sqrt{3} z^2 \\ z^3 \\ \sqrt{3} z \\ 1 \end{pmatrix}$$ **Symplectic section**

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z) = \begin{pmatrix} -\frac{3z+\bar{z}}{2z\bar{z}} & -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} \\ -\frac{\sqrt{3}(z+\bar{z})}{2z\bar{z}^2} & -\frac{z+3\bar{z}}{2z\bar{z}^3} \end{pmatrix}$$ **Matrix $N_{\Lambda\Sigma}$**

SUGRA BH.s = one-dimensional Lagrangian model

Evolution parameter $\tau \sim \frac{1}{r}$ $\qquad \dot{f} \equiv \frac{d}{d\tau} f$

$$\mathcal{L} = \dot{U}^2 + h_{rs}\,\dot{\phi}^r\,\dot{\phi}^s + e^{-2U}\,(\dot{a} + \mathbf{Z}^T \mathbb{C}\dot{\mathbf{Z}})^2 + 2\,e^{-U}\,\dot{\mathbf{Z}}^T\,\mathcal{M}_4\,\dot{\mathbf{Z}}$$

$$\mathcal{L} = \begin{cases} v^2 & > & 0 \\ v^2 & = & 0 \\ -v^2 & < & 0 \end{cases}$$

Time-like geodesic = non-extremal Black Hole

Null-like geodesic = extremal Black Hole

Space-like geodesic = naked singularity

A Lagrangian model can always be turned into a Hamiltonian one by means of standard procedures.

SO BLACK-HOLE PROBLEM = DYNAMICAL SYSTEM

FOR SK_n = symmetric coset space THIS DYNAMICAL SYSTEM is LIOUVILLE INTEGRABLE, always!

OXIDATION 1

The metric

$$ds^2_{(4)} = -e^{U(\tau)}\,(dt + A_{KK})^2 + e^{-U(\tau)}\left[e^{4A(\tau)}\,d\tau^2 + e^{2A(\tau)}\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)\right]$$

where $A_{KK} = 2\,\mathbf{n}\cos\theta\,d\varphi$ → **Taub-NUT charge**

$$\underbrace{\left[e^{-2U}\left(\dot a + Z^\Lambda \dot Z_\Lambda - Z_\Sigma \dot Z^\Sigma\right)\right]}_{\mathbf{n}=\text{Taub NUT charge}}$$

The electromagnetic charges

$$\mathcal{Q}^M = \sqrt{2}\left[e^{-U}\mathcal{M}_4\,\dot Z - \mathbf{n}\,\mathbb{C}\,Z\right]^M = \begin{pmatrix} p^\Lambda \\ e_\Sigma \end{pmatrix}$$

From the σ-model viewpoint all these first integrals of the motion

$$e^{2A(\tau)} = \begin{cases} \dfrac{v^2}{\sinh^2(v\tau)} & \text{if } v^2 > 0 \\ \dfrac{1}{\tau^2} & \text{if } v^2 = 0 \end{cases}$$

→ ***Extremality parameter***

OXIDATION 2

The electromagnetic field-strenghts

$$F^\Lambda = 2\,p^\Lambda \sin\theta\, d\theta \wedge d\varphi + \dot{Z}^\Lambda d\tau \wedge (dt + 2\mathbf{n}\cos\theta\, d\varphi)$$

U, a, $\phi \sim$ z, Z^A *parameterize in the G/H case the coset representative*

$$\mathbb{L}(\Phi) = \exp\left[-a\,L_+^E\right]\,\exp\left[\sqrt{2}\,Z^M\,\mathcal{W}_M\right]\,\mathbb{L}_4(\phi)\,\exp\left[U\,L_0^E\right]$$

gen. in **(2,W)**

Coset repres. in D=4

Element of Borel $(U_{D=3})$

Ehlers SL(2,R)

The Quartic Invariant

The vector of electric and magnetic charges

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Sigma \end{pmatrix} \quad \text{repr. } j = \frac{3}{2}$$ **of SL(2,R)**

Quartic symplectic invariant

$$\mathfrak{I}_4 = \frac{1}{3\sqrt{3}} q_2 p_1^3 + \frac{1}{12} q_1^2 p_1^2 - \frac{1}{2} p_2 q_1 q_2 p_1 - \frac{1}{3\sqrt{3}} p_2 q_1^3 - \frac{1}{4} p_2^2 q_2^2$$

Attraction mechanism & Entropy

$$S_{eff} \equiv \int \mathcal{L}_{eff}(\tau)\, d\tau \quad ;$$

$$\mathcal{L}_{eff}(\tau) = \frac{1}{4}\left(\frac{dU}{d\tau}\right)^2 + g_{ij^\star}\frac{dz^i}{d\tau}\frac{dz^{j^\star}}{d\tau} + e^U V_{BH}(z,\bar{z},\mathcal{Q})$$

$$\begin{aligned} V_{BH}(z,\bar{z},\mathcal{Q}) &= \frac{1}{4}\mathcal{Q}^t \mathcal{M}_4^{-1}(\mathcal{N})\,\mathcal{Q} \\ &= -\frac{1}{2}\left(|Z|^2 + |Z_i|^2\right) \equiv -\frac{1}{2}\left(Z\,\bar{Z} + Z_i g^{i\bar{\jmath}}\,\bar{Z}_{\bar{\jmath}}\right) \end{aligned}$$

Potential

$$\begin{aligned} Z \equiv V^T \mathbb{C}\,\mathcal{Q} &= M_\Sigma\, p^\Sigma - L^\Lambda q_\Lambda \\ Z_i &= \nabla_i Z = U_i \mathbb{C}\,\mathcal{Q} \quad ; \quad Z^{\bar{\jmath}} = g^{\bar{\jmath} i} Z_i \\ \bar{Z}_{\bar{\jmath}} &= \nabla_{\bar{\jmath}} Z = \bar{U}_{\bar{\jmath}} \mathbb{C}\,\mathcal{Q} \quad ; \quad \bar{Z}^i = g^{i\bar{\jmath}}\,\bar{Z}_{\bar{\jmath}} \end{aligned}$$

Central charges of supersym.

Critical Points of the Potential (Ferrara et al)

$$\partial_i V_{BH} = 0 \quad \Rightarrow \quad 0 = 2\, Z_i\, \bar{Z} + \mathrm{i} C_{ijk}\, \bar{Z}^j\, \bar{Z}^k$$

THREE TYPES of Critical Points

$$\begin{array}{lllll} Z_i = 0 \;; & Z \neq 0 \;; & & & \text{BPS attractor} \\ Z_i \neq 0 \;; & Z = 0 \;; & \mathrm{i}\, C_{ijk}\, \bar{Z}^j\, \bar{Z}^k & = 0 & \text{non BPS attractor I} \\ Z_i \neq 0 \;; & Z \neq 0 \;; & \mathrm{i}\, C_{ijk}\, \bar{Z}^j\, \bar{Z}^k & = -2\, Z_i\, \bar{Z} & \text{non BPS attractor II} \end{array}$$

Special Geometry Invariants

$$N_3 \equiv C_{ijk}\, \bar{Z}^i\, \bar{Z}^j\, \bar{Z}^k \quad ; \quad \bar{N}_3 \equiv C_{i^\star j^\star k^\star}\, Z^{i^\star}\, Z^{j^\star}\, Z^{k^\star}$$

$$\begin{array}{lll} i_1 = Z\, \bar{Z} & ; & i_2 = Z_i\, \bar{Z}_{\bar{\jmath}}\, g^{i\bar{\jmath}} \\ i_3 = \frac{1}{6}\left(Z\, N_3 + \bar{Z} \bar{N}_3\right) & ; & i_4 = \mathrm{i}\, \frac{1}{6}\left(Z\, N_3 - \bar{Z} \bar{N}_3\right) \\ i_5 = C_{ijk}\, C_{\bar{\ell}\bar{m}\bar{n}}\, \bar{Z}^j\, \bar{Z}^k\, Z^{\bar{m}}\, Z^{\bar{n}}\, g^{i\bar{\ell}} & ; & \end{array}$$

Invariants at Fixed Points

At BPS attractor points

$$i_1 \neq 0 \quad ; \quad i_2 = i_3 = i_4 = i_5 = 0 \qquad \text{Area}_H = \sqrt{\mathfrak{I}_4}$$

At BPS attractor points of type I

$$i_2 \neq 0 \quad ; \quad i_1 = i_3 = i_4 = i_5 = 0 \qquad \text{Area}_H = \sqrt{-\mathfrak{I}_4}$$

At BPS attractor points of type II

$$i_2 = 3 i_1 \quad ; \quad i_3 = 0 \quad ; \quad i_4 = -2\, i_1^2 \quad ; \quad i_5 = 12\, i_1^2$$

Identity everywhere

$$\mathfrak{I}_4(p,q) = \frac{1}{4}(i_1 - i_2)^2 + i_4 - \frac{1}{4} i_5$$

From coset rep. to Lax equation

$$\Sigma(\tau) \equiv \mathbb{L}^{-1}(\tau)\,\frac{d}{d\tau}\mathbb{L}(\tau)$$

From coset representative

$$\begin{aligned} \Sigma(\tau) &= L(\tau) \oplus W(\tau) \\ W(\tau) \in \mathbb{H}^{\star} &\Rightarrow \eta\, W^T(\tau) + W(\tau)\eta = 0 \\ L(\tau) \in \mathbb{K} &\Rightarrow \eta\, L^T(\tau) - L(\tau)\eta = 0 \end{aligned}$$

decomposition

$$W(\tau) = L_{>}(\tau) - L_{<}(\tau)$$

R-matrix

$$\frac{d}{d\tau}L(\tau) = [W(\tau)\,,\, L(\tau)]$$

Lax equation

Integration algorithm

Initial conditions $L_0 = L(0) \quad , \quad \mathbb{L}_0 = \mathbb{L}(0)$

Building block $\mathcal{C}(\tau) := \exp\left[-2\,\tau\,L_0\right]$

$$\mathfrak{D}_i(\mathcal{C}) := \text{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i}(\tau) \\ \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i}(\tau) \end{pmatrix}, \quad \mathfrak{D}_0(\tau) := 1\,.$$

$$\left(\mathbb{L}(\tau)^{-1}\right)_{ij} \equiv \frac{1}{\sqrt{\mathfrak{D}_i(\mathcal{C})\mathfrak{D}_{i-1}(\mathcal{C})}}\text{Det}\begin{pmatrix} \mathcal{C}_{1,1}(\tau) & \dots & \mathcal{C}_{1,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{1,j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{C}_{i,1}(\tau) & \dots & \mathcal{C}_{i,i-1}(\tau) & (\mathcal{C}(\tau)\mathbb{L}(0)^{-1})_{i,j} \end{pmatrix}$$

Key property of integration algorithm

$$L(\tau) = \mathcal{Q}(\mathcal{C})\, L_0 \left(\mathcal{Q}(\mathcal{C})\right)^{-1}$$

$$\mathcal{Q}(\mathcal{C}) \in \mathsf{H}^\star$$

Hence all LAX evolutions occur within distinct orbits of H*

Fundamental Problem: classification of ORBITS

The role of H*

$$\mathsf{U}_{D=3} \supset \begin{cases} \mathsf{H} \\ \mathsf{and} \\ \mathsf{H}^\star \end{cases}$$

Max. comp. subgroup — **COSMOL.**

Different real form of H — **BLACK HOLES**

In our simple $G_{2(2)}$ model

$$\mathbb{H}^\star = \mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$$

The algebraic structure of Lax

For the simplest model ,the Lax operator, is in the representation

$$\left(j=\frac{1}{2}\right)\times\left(j=\frac{3}{2}\right)$$

of $\mathfrak{sl}(2,R)\times\mathfrak{sl}(2,R)$

$$L \sim \Delta^{\alpha|A}$$

We can construct invariants and tensors with powers of L

Invariants & Tensors

$$\mathfrak{h}_6 = \frac{1}{6}\mathrm{Tr}L^6 + \frac{1}{96}\left(\mathrm{Tr}L^2\right)^3$$

$$\mathfrak{h}_2 = \frac{1}{4}\mathrm{Tr}\,L^2$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{symm} = \underbrace{(\mathbf{j}=\mathbf{3})}_{7}\oplus\underbrace{(\mathbf{j}=\mathbf{1})}_{1}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{antisym} = \underbrace{(\mathbf{j}=\mathbf{2})}_{5}\oplus\underbrace{(\mathbf{j}=\mathbf{0})}_{1}$$

Quadratic Tensor

$$\mathcal{T}^{xy} \equiv \frac{128}{\sqrt{3}}\, t^{xy}{}_{AB}\,\Delta^{\alpha|A}\,\Delta^{\beta|B}\,\epsilon_{\alpha\beta}$$

Tensors 2

QUADRATIC
BIVECTOR

$$\mathcal{W}^{a|x} \equiv 1280\, \Pi^{a}_{\alpha\beta}\, \Sigma^{z}_{AB}\, \Delta^{\alpha|A}\, \Delta^{\beta|B}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{symm} = \underbrace{(\mathbf{j}=\mathbf{3})}_{7}\oplus\underbrace{(\mathbf{j}=\mathbf{1})}_{1}$$

$$\left[\left(\mathrm{j}=\frac{3}{2}\right)\otimes\left(\mathrm{j}=\frac{3}{2}\right)\right]_{antisym} = \underbrace{(\mathbf{j}=\mathbf{2})}_{5}\oplus\underbrace{(\mathbf{j}=\mathbf{0})}_{1}$$

Tensors 3

Hence we are able to construct quartic tensors

$$\mathfrak{T}^{xy} = \mathcal{W}^{a|x}\,\mathcal{W}^{b|y}\,\eta_{ab}$$

$$\mathbb{T}^{ab} = \mathcal{W}^{a|x}\,\mathcal{W}^{b|y}\,\eta_{xy}$$

ALL TENSORS, QUADRATIC and QUARTIC are symmetric

Their signatures classify orbits, both regular and nilpotent!

Tensor classification of orbits

Orbit	Order Nilp.	Stand. Repr.	Stab. subg.	Sign. $\mathcal{T}^{xy}$	Sign. $\mathfrak{T}^{xy}$	Sign. $\mathbb{T}^{ab}$	Bivect. $W^{a\|x}$	$\mathfrak{I}_4$ at $\mathbf{n}=0$	Dim. $\mathbf{n}=0$ shell
Schw.	∞	$\mathfrak{S}$	$O(2)$	$\{+,+,+\}$	$\{+,0,0\}$	$\{+,0,0\}$	$\neq 0$	$\neq 0$	4
Dil.	∞	$\mathfrak{D}$	$O(1,1)$	$\{-,-,+\}$	$\{-,0,0\}$	$\{-,0,0\}$	$\neq 0$	$\neq 0$	4
N01	2	$\mathcal{L}^{NO_1}$	$O(1,1) \ltimes \mathbb{R}^2$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,0\}$	0	0	2
N02	3	$\mathcal{L}^{NO_2}$	$O(1,1) \ltimes \mathbb{R}$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,0\}$	$\neq 0$	0	3
N03	3	$\mathcal{L}^{NO_3}$	$\mathbb{R}$	$\{0,0,\bullet\}$	$\{0,0,0\}$	$\{0,0,0\}$	$\neq 0$	< 0	4
N04	3	$\mathcal{L}^{NO_4}$	$\mathbb{R}$	$\{0,0,0\}$	$\{0,0,0\}$	$\{0,0,\bullet\}$	$\neq 0$	> 0	4
N05	7	$\mathcal{L}^{NO_5}$	0	$\{0,+,-\}$	$\{0,0,-\}$	$\{0,+,-\}$	$\neq 0$	< 0	5

How do we get to this classification? The answer is the following: by choosing a new Cartan subalgebra inside H* and recalculating the step operators associated with roots in the new Cartan Weyl basis!

Relation between old and new Cartan Weyl bases

New Cartan Weyl generators	their form in the HK-basis
H_1	h_3
H_2	h_5
E_1	$\frac{1}{2\sqrt{2}}(k_3 - 3\mathcal{H}_1 - \mathcal{H}_2)$
E_2	$-\frac{1}{4}\sqrt{\frac{3}{2}}(k_1 + k_2 + k_4 - k_6)$
E_3	$\frac{1}{4\sqrt{2}}(-3h_1 + h_2 - h_4 + 3h_6)$
E_4	$\frac{1}{4\sqrt{2}}(-3k_1 + k_2 + k_4 + 3k_6)$
E_5	$\frac{1}{4}\sqrt{\frac{3}{2}}(-h_1 - h_2 + h_4 + h_6)$
E_6	$\frac{1}{2}\sqrt{\frac{3}{2}}(k_5 - \mathcal{H}_1 - \mathcal{H}_2)$

$\in \mathbb{K}$

$$
\begin{aligned}
h_1 &= e_2 + f_2 & k_1 &= e_2 - f_2 \\
h_2 &= e_1 - f_1 & k_2 &= e_1 + f_1 \\
h_3 &= e_3 + f_3 & k_3 &= e_3 - f_3 \\
h_4 &= e_4 + f_4 & k_4 &= e_4 - f_4 \\
h_5 &= e_5 + f_5 & k_5 &= e_5 - f_5 \\
h_6 &= e_6 - f_6 & k_6 &= e_6 + f_6
\end{aligned}
$$

Hence we can easily find nilpotent orbits

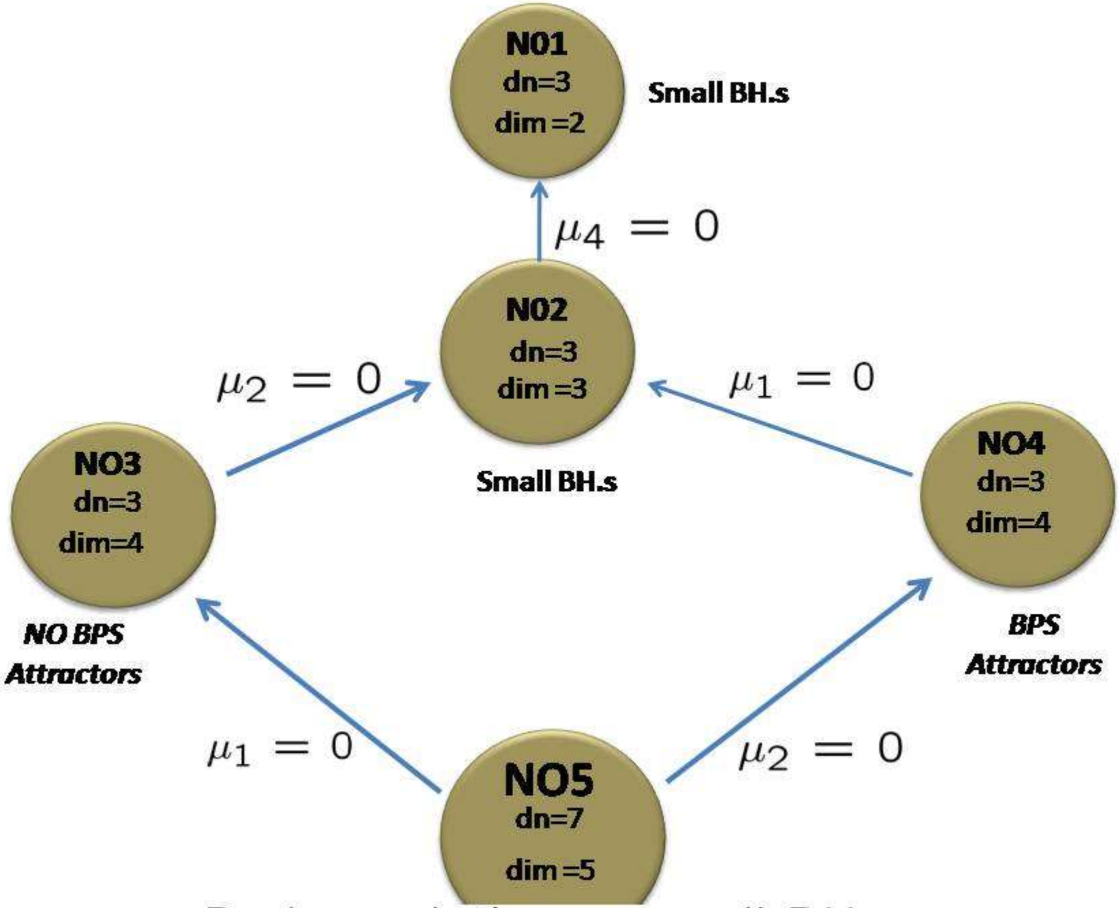
Every orbit possesses a representative of the form

$$\mathcal{O}_{\vec{\mu}} = \mu_1 E_1 + \mu_2 E_2 + \mu_4 E_4 + \mu_6 E_6$$

Generic nilpotency 7. Then imposereduction of nilpotency

Orbit name	Abstract Repres.	Repr. at Taub-NUT $= 0$		
NO_1	$\mu_2 E_2 + \mu_6 E_6$	$\mathcal{L}^{NO_1}$	$=$	$2\sqrt{\frac{2}{3}}\mathcal{R}_1 E_2 \mathcal{R}_1^{-1}$
NO_2	$\mu_4 E_4 + \mu_6 E_6$	$\mathcal{L}^{NO_2}$	$=$	$4\sqrt{2}\mathcal{R}_2 E_4 \mathcal{R}_2^{-1}$
NO_3	$\mu_2 E_2 + \mu_4 E_4 + \mu_6 E_6$	$\mathcal{L}^{NO_3}$	$=$	$\mathfrak{L}_0^{(p\|q)}$
NO_4	$\mu_1 E_1 + \mu_4 E_4 + \mu_6 E_6$	$\mathcal{L}^{NO_4}$	$=$	$\widehat{\mathfrak{L}}_0^{(1\|-1)}$
NO_5	$\mu_1 E_1 + \mu_2 E_2 + \mu_4 E_4 + \mu_6 E_6$	$\mathcal{L}^{NO_5}$	$=$	$2\sqrt{\frac{2}{3}}\mathcal{R}_3 (E_1 + E_2) \mathcal{R}_3^{-1}$

The general pattern



Broken solutions $\sim$ small BH.s

EXAMPLE : NON BPS attractor with 2 charges

Initial Lax

$$L_0 =$$

$$\begin{pmatrix}
q\sqrt{\kappa} & \frac{q\xi}{\sqrt{\kappa}} & -\frac{q\xi}{\sqrt{\kappa}} & \frac{q\sqrt{\kappa}}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{q\xi}{\sqrt{\kappa}} & \frac{p+q(\kappa^2+3\xi^2)}{2\kappa^{3/2}} & \frac{-3q\xi^2-p}{2\kappa^{3/2}} & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & -\frac{q\sqrt{\kappa}}{2} & 0 & 0 \\
\frac{q\xi}{\sqrt{\kappa}} & \frac{3q\xi^2+p}{2\kappa^{3/2}} & \frac{q(\kappa^2-3\xi^2)-p}{2\kappa^{3/2}} & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & 0 & \frac{q\sqrt{\kappa}}{2} & 0 \\
-\frac{q\sqrt{\kappa}}{\sqrt{2}} & -\frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & 0 & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & -\frac{q\sqrt{\kappa}}{\sqrt{2}} \\
0 & \frac{q\sqrt{\kappa}}{2} & 0 & \frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & \frac{-q\kappa^2+3q\xi^2+p}{2\kappa^{3/2}} & \frac{3q\xi^2+p}{2\kappa^{3/2}} & -\frac{q\xi}{\sqrt{\kappa}} \\
0 & 0 & -\frac{q\sqrt{\kappa}}{2} & -\frac{\sqrt{2}q\xi}{\sqrt{\kappa}} & \frac{-3q\xi^2-p}{2\kappa^{3/2}} & \frac{-q\kappa^2-3q\xi^2-p}{2\kappa^{3/2}} & \frac{q\xi}{\sqrt{\kappa}} \\
0 & 0 & 0 & \frac{q\sqrt{\kappa}}{\sqrt{2}} & \frac{q\xi}{\sqrt{\kappa}} & \frac{q\xi}{\sqrt{\kappa}} & -q\sqrt{\kappa}
\end{pmatrix}$$

p,q charges $z(\infty) = \xi + \mathrm{i}\,\kappa$

Initial coset representative

$$\mathbf{L}_0^{(\xi|\kappa)} = \begin{pmatrix} \sqrt{\kappa} & \frac{\xi}{\sqrt{\kappa}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\kappa}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa & \sqrt{2}\xi & \frac{\xi^2}{\kappa} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{\sqrt{2}\xi}{\kappa} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\kappa} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\kappa} & \frac{\xi}{\sqrt{\kappa}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\kappa}} \end{pmatrix}$$

Solution

$$\exp[U(\tau)] = \frac{\kappa^{3/4}}{\sqrt{-q^3\kappa^3\tau^3 - q^3\kappa\xi^2\tau^3 + 3q^2\kappa^{5/2}\tau^2 + 3q^2\sqrt{\kappa}\xi^2\tau^2 + p(q\sqrt{\kappa}\tau - 1)^3\tau - 3q\kappa^2\tau - 3q\xi^2\tau + \kappa^{3/2}}}$$

$$\mathrm{Im}\, z(\tau) = \frac{\sqrt[4]{\kappa}\sqrt{-q^3\kappa^3\tau^3 - q^2\kappa(q\xi^2+3p)\tau^3 + 3q^2\kappa^{5/2}\tau^2 + 3q\sqrt{\kappa}(q\xi^2+p)\tau^2 - 3q\kappa^2\tau - (3q\xi^2+p)\tau + \kappa^{3/2}(pq^3\tau^4+1)}}{(q\sqrt{\kappa}\tau - 1)^2}$$

$$\mathrm{Re}\, z(\tau) = \frac{\xi}{(q\sqrt{\kappa}\tau - 1)^2}$$

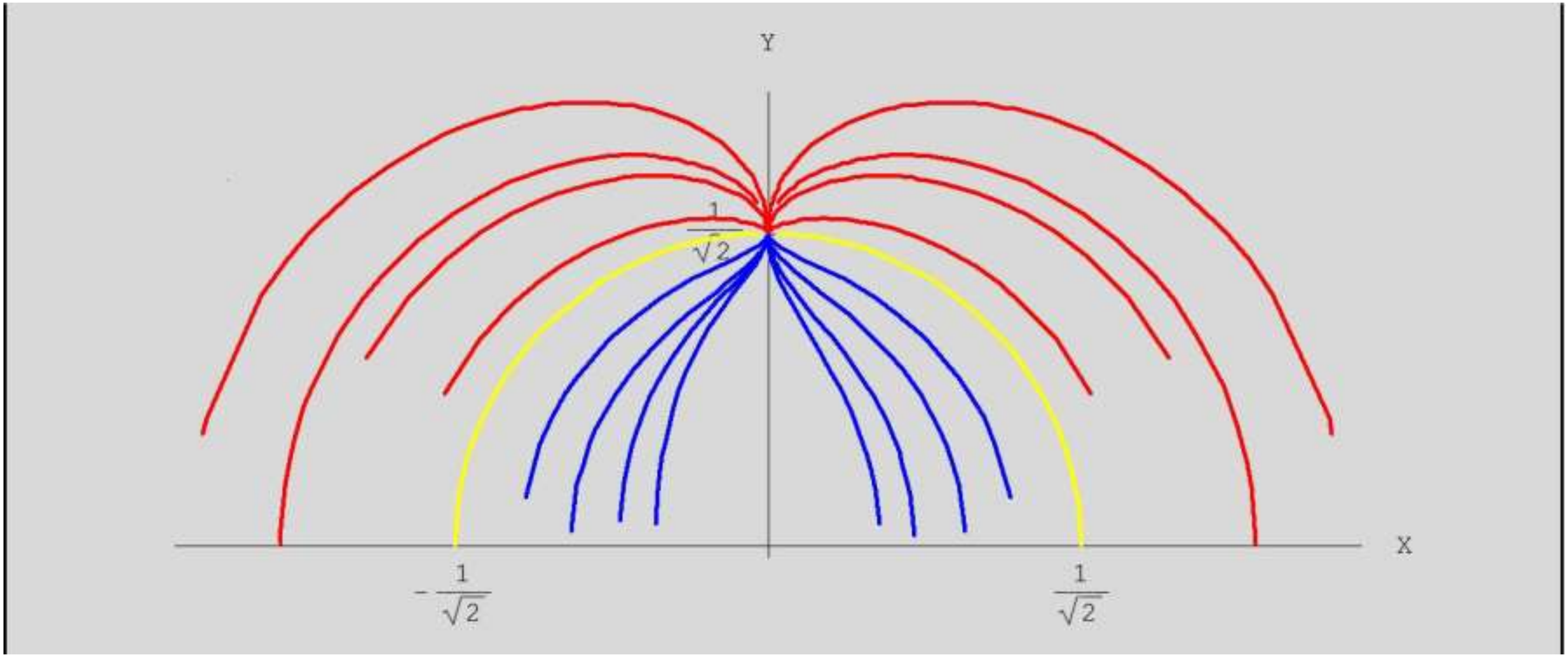
At the horizon

$$\frac{1}{4\pi}\mathrm{Area}_H \equiv r_H^2 = \lim_{\tau \to -\infty} \frac{1}{\tau^2}\exp[-U(\tau)] = \sqrt{p\,q^3}$$

$$\lim_{\tau \to -\infty} z(\tau) = \mathrm{i}\,\frac{pq}{\sqrt{p\,q^3}} = \mathrm{i}\sqrt{\frac{p}{q}}$$

$$p\,q^3 = -\mathfrak{I}_4$$

The attraction mechanism in picture



Liouville Integrability 1°

The Poissonian structure

$$\mathfrak{B} = \sum_{A=1}^{8} \Phi^A T_A \in \mathrm{Borel}\left(\mathfrak{g}_{2(2)}\right) \quad ; \quad L = \mathfrak{B} + \eta \, \mathfrak{B}^T \eta$$

$$[T_A \, , \, T_B] = f_{AB}{}^{C} \, T_C$$ **Borel subalgebra**

$$\{F \, , \, G\} \equiv \frac{\partial F}{\partial \Phi^A} \frac{\partial G}{\partial \Phi^B} f^{AB}{}_{C} \, \Phi^C$$ **Poisson Bracket**

$$\mathfrak{H}_{quad} = \mathrm{cost} \, \mathrm{g}_{AB} \, \Phi^A \, \Phi^B$$

Evolution equations

$$\frac{d}{d\tau} \Phi^A = \left\{ \mathfrak{H}_{quad} \, , \, \Phi^A \right\}$$

Liouville Integrability 2°

Kostant Decomposition

For any Lie algebra element holds true

$$\mathcal{B}\, g\, \mathcal{B}^{-1} = \sum_{\alpha>0} K_\alpha(g)\, E^{\alpha} + \sum_{i=1}^{r} E^{-\alpha^i_{orth}}$$

$$\alpha^r_{orth} > \alpha^{r-1}_{orth} > \ldots > \alpha^1_{orth} \quad \Big| \quad \left(\alpha^i_{orth}, \alpha^j_{orth}\right) = 0 \quad \text{if } i \neq j$$

$$K_\alpha(Lax) = \text{const. of motion}$$

How to find commuting hamiltonians?

The involutive hamiltonians

$$\mathcal{P}\left(\lambda, \mu_1, \ldots \mu_{r-1}\right) \equiv \mathrm{Det}\left(\mathcal{KN}(L) - \sum_{i=1}^{r-1} \mu_i \, E^{\alpha^{r+1-i}_{orth}} - \lambda \, \mathbf{1}\right)$$

Kostant normal form of Lax

For G_{2(2)}

$$\mathcal{P}\left(\lambda, \mu\right) = \lambda^7 + \mathfrak{K}_1 \lambda^5 + \mathfrak{K}_1^2 \lambda^3 + \mathfrak{K}_2 \lambda - 3\lambda^5 \mu + \mathfrak{K}_1 \lambda^3 \mu + \mathfrak{K}_3 \lambda \mu - \tfrac{9}{4} \lambda^3 \mu^2 + \mathfrak{K}_4 \lambda \mu^2$$

$$\left\{\mathfrak{K}_i \, , \, \mathfrak{K}_j\right\} = 0$$